Some Special Equations

LEGENDRE'S EQUATION

The differential equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$$

where n is a fixed real number, is known as Legendre's equation.

Checking for singular points, we have

$$P(x) = \frac{-2x}{\left(1-x^2\right)}$$

$$Q(x) = \frac{n(n+1)}{\left(1-x^2\right)}$$

Therefore the equation has a singular points at $x = \pm 1$. The point at $x = 1$ is a regular singular point.

Hence, a series solution for

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$$

about $x = 1$ can be obtained by the method of Frobenius.
Set $z = x - 1$, the equation is transformed into

$$z(z+2)y'' + 2(z+1)y' - n(n+1)y = 0$$

In this form we have

$$\lim_{z \to 0} zP(x) = \frac{2z(z+1)}{z(z+2)} = 1 = p_0$$

$$\lim_{z \to 0} z^2 Q(x) = \lim_{z \to 0} \frac{-z^2 n(n+1)}{z(z+2)} = 0 = q_0$$

The indicial equation for the series solution

$$y(z) = \sum_{k=0}^{\infty} a_n z^{k+r}$$

is therefore

$$r^2 + (1-1)r + 0 = r^2 = 0$$

and has roots $r_1 = r_2 = 0$.
Substituting $y(x)$ into the equation leads to

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{k!(1)_k} \left(\frac{1-x}{2}\right)^k$$

Here y_1 is expressed in terms of the orginal variable x and the value $a_0 = 1$ has been used.

For n a nonnegative integer, the factor

$$(-n)_k = (-n)(-n+1)(-n+2)\cdots(-n+k-1)$$

will be zero for $k \geq n+1$.
Hence,

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{k!(1)_k} \left(\frac{1-x}{2}\right)^k$$

is a polynomial of degree n. Also, $y_1(1) = 1$

These polynomial solutions of $\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$ are called Legendre polynomials and are denoted by $P_n(x)$:

$$P_n(x) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{k!(1)_k} \left(\frac{1-x}{2}\right)^k$$

Legendre polynomials satisfy the recurrence formula:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

and also Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(\left(x^2-1\right)^n\right)$$

Chebyshev's Equation

"To isolate mathematics from the practical demands of the sciences is to invite the sterility of a cow shut away from the bulls" - Pafnuty Chebyshev

-Pafnuty Chebyshev lived 1821-1894

-Researched probability theory, quadratic forms, orthogonal functions, the theory of integrals, the construction of maps, and the calculation of geometric volumes

The Differential equation

$$\left(1-x^2\right)y'' - xy' + n^2 y = 0$$

is known as Chebyshev's equation.
The polynomials that are solutions to this equation satisfy the recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$
$$T_0(x) = 1$$
$$T_1(x) = x$$

Therefore, it is clear that:

$$\begin{aligned} T_2 &= 2(x)T_1(x) - T_0(x) \\ &= 2x^2 - 1 \end{aligned}$$

$$\begin{aligned} T_3(x) &= 2(x)T_2(x) - T_1(x) \\ &= 4x^3 - 2x - x \\ &= 4x^3 - 3x \end{aligned}$$

$$\begin{aligned} T_4(x) &= 2(x)T_3(x) - T_2(x) \\ &= 8x^4 - 6x^2 - 2x^2 + 1 \\ &= 8x^4 - 8x^2 + 1 \end{aligned}$$

-In general the generating function for $T_n(x)$ can be expressed by:

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n$$

-The Chebyshev polynomials also carry some interesting properties:

$$T_n(1) = 1$$

$$T_n(-1) = -1^n$$

$$T_{2n}(0) = (-1)^n$$

-For $x = \cos(\theta)$

$$T_n(x) = T_n(\cos(\theta)) = \cos(n\theta) = T_{-n}(x)$$

-Chebyshev polynomials are also used in the study of orthogonal series and they are used in relation to Taylor Series

Hermite Polynomials

The Hermite Polynomials arise as solutions to the Hermite equation

$$y'' - 2xy' + 2ny = 0$$

The polynomials have a generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

where $H_n(x)$ is defined by its Rodrigues formula

$$(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

The Hermite equation comes from a differential equation that describes the linear harmonic equation in quantum physics

$$\frac{d^2w}{dx^2} + \left(2p + 1 - x^2\right)w = 0$$

This becomes the Hermite DE after making the substitution

$$w = ye^{-\frac{x^2}{2}}$$

As we know, the Hermite equation has a recurance relationship for the coeficients after using the method of series solution

$$a_{n+2} = -\frac{2(p-n)}{(n+1)(n+2)}a_n$$

One interesting thing to do is to expand the degree of the equation to an arbitrary complex value for *n*. The way this is done is to use Rodrigues' equation

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Notice that the only thing holding us back from extending the DE is the derivative term. To fix this problem, we use the definition of the differintegral of arbitrary order $D_t^\alpha f(t)$

$$\frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$

where n is the integer floor of α, the order of differintegration and τ is a dummy variable. Substituting into the Rodrigues' equation and simplifying, we get

$$H_\alpha(x) = (-1)^\alpha e^{x^2} \frac{1}{\Gamma(\alpha - n)} \int_a^x \frac{\frac{d^n}{d\tau^n} e^{-\tau^2} d\tau}{(x-\tau)^{\alpha-n+1}}$$