MA651 Topology. Lecture 1. Elements of Set Theory 1.

This text is based on the book "Topology" by James Dugundgji

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

1 Basic Notation

Symbolic logic notation.

Definition 1.1. If p and q are propositions, then:

$p \vee q$ (read: p or q) denotes the disjunction of p and q. The assertion "$p \vee q$" is true whenever at least one of p,q is true.

$p \wedge q$ (read: p and q) denotes the conjunction of p and q. The assertion "$p \wedge q$" is true only in case both p and q are true.

$\neg\, q$ (read: not q) denotes the negation of p. The assertion "$\neg\, q$" is true only if q is false.

$p \Rightarrow q$ is read: p implies q. By definition, "$p \Rightarrow q$" denotes "$(\neg p) \vee q$". In particular, the statement "$p \Rightarrow q$" is true if and only if the statement "$(\neg\, q) \Rightarrow (\neg\, p)$" is true.

$p \Leftrightarrow q$ is read: p is logically equivalent to q. By definition, "$p \Leftrightarrow q$" denotes "$(p \Rightarrow q) \wedge (q \Rightarrow p)$".

An expression $p(x)$ that becomes a proposition whenever values from a specified domain of discourse are substituted for x is called a propositional function or, equivalently, a condition on x; and p is called a property, or predicate. The assertion "y has property p" means that "$p(y)$" is true. Thus, if $p(x)$ is the proportional function "x is an integer", then p denotes the property "is an integer", and "$p(2)$" is true whereas "$p(1/2)$" is false.

The quantifier "there exists" is denoted by $\exists$, an the quantifier "for each" is denoted by $\forall$. The assertion "$\forall x\ \exists y\ \forall z : p(x, y, z)$" reads "For each x there exists a y such that for each z, $p(x, y, z)$ is true"; its negation is obtained mechanically by changing the sense of each quantifier (preserving the given order of the variables!) and negating the proposition: thus, "$\exists x\ \forall y\ \exists x : \neg\, p(x, y, z)$".

2 Sets

Intuitively, we think of a set as something made up by all the objects that satisfy some given conditions, such as the set of prime numbers, the set of points on a line, or the set of objects named in a given list. The objects making up the set are called the elements, or members, of the set and may themselves be sets, as in the set of all lines in the plane.

Rigorously, the word *set* is an undefined term in mathematics, so that definite axioms are required to govern the use of this term. Although we shall deal with sets on an intuitive basis until we discuss an axiom systems, whenever we apply the label *set* to something, we shall later provide this usage to have been formally justified.

The membership relation is denoted by $\in$ and sets are generally indicated by capital letters: "$a \in A$" is read "a belongs to (is a member, element, point of) the set A"; $\neg(a \in A)$ is written $a \notin A$. The notation $a = b$ will mean that the objects a and b are the same, and $a \neq b$ denotes $\neg(a = b)$. If A, B sets, then $A = B$ will indicate that A and B have the same elements; that is, $\forall x : (x \in A) \Leftrightarrow (x \in B)$; $\neg(A = B)$ is written $A \neq B$.

$A \subset B$ (or $B \supset A$), read "A is a subset of (is contained in) B", signifies that each element of A is an element of B, that is $\forall x : (x \in A) \Rightarrow (x \in B)$; equality is *not* excluded - we call A a *proper* subset of B ($A \subsetneq B$ whenever $(A \subset B) \wedge (A \neq B)$. The relations $\subset$ and $\subsetneq$ are called *inclusion* and *proper inclusion*, respectively. the following statements are evident:

Proposition 2.1. $A \subset A$ for each set A.

Proposition 2.2. If $A \subset B$ and $B \subset C$, then $A \subset C$ (that is, C is transitive).

Proposition 2.3. $A = B$ if and only if both $A \subset B$ and $B \subset A$.

Of these, the last statement is very important: the equality of two sets is usually proved by showing each of the two inclusions valid.

The axioms of set theory allow only two methods for forming subsets of a given set. One of these is by appeal to the axiom of choice, and will be discussed later. The other is by use of the following principle: If A is a set and p is a property that each element of A either has or does not have, then all the $x \in A$ having the property p from a set. This subset of A is denoted by $\{x \in A \mid p(x)\}$; it is uniquely determined by the property p. Clearly, $\{x \in A \mid p(x)\} \subset \{x \in A \mid q(x)\}$ if and only if $\forall x \in A : p(x) \Rightarrow q(x)$; thus two properties determine the same subset of A whenever each object in A having one of them also has the other.

Example 2.1. *If A is the real line, the closed unit interval is* $\{x \in A \mid 0 \leqslant x \leqslant 1\}$

Example 2.2. *If A is the real line, $\{x \in A \mid x^2 = 1\} = \{x \in A \mid x^4 = 1\}$ even thought the defining properties are different. Note that if A was the set of complex numbers, these two properties would not determine the same subset.*

Example 2.3. *For each set A, $\{x \in A \mid x = x\} = \{x \in A \mid x \in A\} = A$*

For each set A, the null subset $\emptyset_A \subset A$ is $\{x \in A \mid x \neq x\}$; it has no members, since each $x \in A$ satisfies $x = x$.

Proposition 2.4. All null subsets are equal. Thus there is one and only one null set, $\emptyset$, and it is contained in every set: $\emptyset \subset A$ for every set A

Proof. Let A, B be any two sets. If $\emptyset_A \subset \emptyset_B$ were false, these would be at least one element a in $\emptyset_A$ not in $\emptyset_B$; in particular, we would then have as an $a \in A$ such that $a \neq a$ and this is impossible. In the same way. $\emptyset_B \subset \emptyset_A$; therefore, by (2.3), $\emptyset_A = \emptyset_B$ and all null sets are equal. $\square$

3 Boolean Algebra

Definition 3.1. Let Γ be a given set, and A, B two subsets. The *union*, $A \bigcup B$, of A and B is $\{x \in \Gamma \mid x$ belongs to at least one of $A, B\}$. The *intersection*, $A \bigcap B$, of A and B is $\{x \in \Gamma \mid x$ belongs to both A and $B\}$.

According to the definition, a necessary and sufficient condition for two sets A, B to have elements in common is that $A \bigcap B \neq \emptyset$; if $A \bigcap B = \emptyset$, the sets A and B are called *disjoint*. The following two statements are immediate consequences of (3.1):

Proposition 3.1. For any two sets A, B, always $A \bigcap B \subset A \subset A \bigcup B$

Proposition 3.2. If $A \subset C$ and $B \subset D$, then $A \bigcup B \subset C \bigcup D$ and $A \bigcap B \subset C \bigcap D$

The formal properties of the operations $\bigcup$ and $\bigcap$ are given in

Theorem 3.1. *Each of the operations $\bigcup$ and $\bigcap$ is*

1. *Idempotent: $\forall A : A \bigcup A = A = A \bigcap A$.*
2. *Associative: $A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C$ and $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$*
3. *Commutative: $A \bigcup B = B \bigcup A$ and $A \bigcap B = B \bigcap A$*

Furthermore, $\bigcap$ distributes over $\bigcup$ and $\bigcup$ distributes over $\bigcap$:

$$A\bigcap(B\bigcup C) = (A\bigcap B)\bigcup(A\bigcap C),$$

$$A\bigcup(B\bigcap C) = (A\bigcup B)\bigcap(A\bigcup C).$$

Proof. Verification of (1) – (3) is trivial. To give an example of a set-theoretic proof, we establish distributivity of $\bigcap$ over $\bigcup$. Using (2.3), we find that the proof decomposes into two parts:

(a) Left side $\subset$ right side:

$$\begin{aligned} x \in A\bigcap(B\bigcup C) &\Rightarrow (x \in A) \wedge [(x \in B) \vee (x \in C)] \\ &\Rightarrow (x \in A\bigcap B) \vee (x \in A\bigcap C) \\ &\Rightarrow x \in (A\bigcap B)\bigcup(A\bigcap C). \end{aligned}$$

(b) Right side $\subset$ left side: All implications in the above string are reversible.

□

Because of associativity, we can designate $A\bigcup(B\bigcap C)$ simply by $A\bigcup B\bigcup C$. Similarly, a union (or intersection) of four sets, say $(A\bigcup B)\bigcup(D\bigcup C)$, can be written $A\bigcup B\bigcup C\bigcup D$ because, by associativity, the distribution of parentheses is irrelevant, and by commutativity, the order of the terms plays no role. By induction, the same remarks apply to the union (or intersection) of any finite number of sets. The union of n sets, $A_1, \cdots, A_n$ is written $\bigcup\limits_1^n A_i$.

The relation between $\bigcap$, $\bigcup$, and $\subset$ is given in

Proposition 3.3. The statements (1) $A \subset B$, (2) $A = A\bigcap B$, and (3) $B = A\bigcup B$ are equivalent.

Proof. We prove only (1) $\Leftrightarrow$ (2) leaving the rest as a homework. If (1), then we have

$$A = A\bigcap A \subset A\bigcap B \subset A,$$

which proves (2). Conversely, if (2), then $A = A\bigcap B \subset B$, establishing (1). □

Definition 3.2. The difference $A - B$ of two sets is $\{x \in A \mid x \notin B\}$

Example 3.1. *If* $A = [0, 1]$ *and* $B = [1, 2]$*, then* $A - B = A$

Example 3.2. *If* $A - \emptyset = A$ *and* $A - B = A - (A\bigcap B)$.

The difference operator does not have formal properties so simple as those of $\bigcup$ and $\bigcap$: for example, since $(A \bigcup A) - A \neq A \bigcup (A - A)$, the location of parentheses in $A \bigcup A - A$ is important. To construct a suitable calculus involving the difference operator, we introduce the complementation operation:

Definition 3.3. If $B \subset A$, the complement $\mathbf{C}_A B$ of B with respect to A is $A - B$.

Note that the complementation operation is defined *only* when one set is contained in the other, whereas the difference operation does not have such a restriction. The relation between (3.2) and (3.3) is given in

Proposition 3.4. For any two sets A, B if the compliment is taken with respect to any set E containing $A \bigcup B$, then $A - B = A \bigcap \mathbf{C}_E B$.

Proof. Since $A \bigcup B \subset E$ we have

$$\begin{aligned} A - B &= \{x \in E \mid (x \in A) \vee (x \notin B)\} \\ &= \{x \in E \mid x \in A\} \bigcap \{x \in E \mid x \notin B\} = A \bigcap \mathbf{C}_E B. \end{aligned}$$

□

The following properties of complementation are immediate:

Proposition 3.5. If E any set containing $A \bigcup B$, then:

1. $A \bigcap \mathbf{C}_E A = \emptyset, \quad A \bigcup \mathbf{C}_E A = E$
2. $\mathbf{C}_E(\mathbf{C}_E A) = A$
3. $\mathbf{C}_E \emptyset = E, \quad \mathbf{C}_E E = \emptyset$
4. $A \subset B$ if and only if $\mathbf{C}_E B \subset \mathbf{C}_E A$

We write $\mathbf{C}$ instead of $\mathbf{C}_E$ whenever the set E has been specified and is to be kept fixed. The basic relation between $\bigcup$, $\bigcap$, and $\mathbf{C}$ is

Theorem 3.2. ***(De Morgan)*** *If complements be taken with respect to any set E containing $A \bigcup B$, then:*

1. $\mathbf{C}(A \bigcup B) = (\mathbf{C}A) \bigcap (\mathbf{C}B)$

2. $\mathbf{C}(A \bigcap B) = (\mathbf{C}A) \bigcup (\mathbf{C}B)$

Proof. We have

$$\begin{aligned} x \in \mathbf{C}(A \bigcup B) &\Leftrightarrow x \notin A \bigcup B \\ &\Leftrightarrow (x \notin A) \wedge (x \notin B) \\ &\Leftrightarrow x \in (\mathbf{C}A) \bigcap (\mathbf{C}B) \end{aligned}$$

and this establishes 1. The proof of 2 is similar; it can, however, be deducted from 1 by "complements":

$$\mathbf{C}[\mathbf{C}A \bigcup \mathbf{C}B] = \mathbf{C}\mathbf{C}A \bigcap \mathbf{C}\mathbf{C}B = A \bigcap B,$$

and apply Proposition (3.5) 2. □

Remark: the formulas in Propositions (3.1), (3.2), (3.3), (3.4), (3.5), and Theorems (3.2), (3.1) comprise a short list of results useful for formal calculations with sets; the role of Proposition (3.4) is to change differences to complements. As examples, we prove:

Example 3.3. $(A - B) \bigcup (B - A) = (A \bigcup B) - (A \bigcap B)$. *For taking complements with respect to* $E = A \bigcup B$, *and using Theorem (3.1), (3.3), (3.5), Theorem (3.2), gives*

$$\begin{aligned} (A - B) \bigcup (B - A) &= (A \bigcap \mathbf{C}B) \bigcup (B \bigcap \mathbf{C}A) \\ &= (A \bigcup B) \bigcap (A \bigcup \mathbf{C}A) \bigcap (\mathbf{C}B \bigcup B) \bigcap (\mathbf{C}B \bigcup \mathbf{C}A) \\ &= A \bigcup B \bigcap \mathbf{C}(B \bigcap A) = (A \bigcup B) - (A \bigcap B) \end{aligned}$$

Example 3.4. *If* $A \bigcup X = E$ *and* $A \bigcap X = \emptyset$, *then* $X = \mathbf{C}_E A$. *For,*

$$\begin{aligned} X = E \bigcap X &= (A \bigcup \mathbf{C}_E A) \bigcap X = X \bigcap \mathbf{C}_E A \\ &= (A \bigcap \mathbf{C}_E A) \bigcup (X \bigcap \mathbf{C}_E A) = (A \bigcup X) \bigcap \mathbf{C}_E A \\ &= E \bigcap \mathbf{C}_E A = \mathbf{C}_E A \end{aligned}$$

Remark: A Boolean algebra B is a set together with two binary operations $+$, $\cdot$, and a unary operation $'$, satisfying the following axioms:

1. Each operation $+, \cdot$ is commutative.

2. There exist elements 0, 1 with $a + 0 = a$, $a \cdot 1 = a$ for every $a \in B$.

3. The distributive law

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c, \\ a + (b \cdot c) &= (a + b) \cdot (a + c) \end{aligned}$$

   hold.

4. $a \cdot a' = 0$ and $a + a' = 1$ for each $a \in B$.

(it is not necessary to postulate associativity of the $+$ and $\cdot$ operations; this is a consequence of the axioms.) The collection of all subsets of a fixed set E, with $+$, $\cdot$, $'$, 0, 1 interpreted as $\bigcup$, $\bigcap$, $\mathbf{C}_E$, $\emptyset$, E, respectively, evidently form a Boolean algebra. By observing that the systematic interchange of $+$ with $\cdot$ and 0 with 1 in the axioms simply gives the same set of axioms, we obtain the duality principle: For each formula true in a Boolean algebra, there is a "dual" true formula obtained by replacing each occurrence of $+$, $\cdot$, 0, 1 with $\cdot$, $+$, 1, 0, respectively. This is the "method of complements"; observe that each one of De Morgan's rules follows from the other by duality.

The theory of Boolean algebras is equivalent to that of commutative rings with unit, in which each element is idempotent, $a \cdot a = a$, (that is, Boolean rings). Indeed, given a Boolean algebra B, define operators $\bigoplus$, $\bigodot$ by $a \bigoplus b = (a \cdot b') + (a' \cdot b)$, $a \bigodot b = a \cdot b$; with $\bigoplus$, $\bigodot$, B is a Boolean ring, $r(B)$. Conversely, from a Boolean ring R, one obtains a Boolean algebra $b(R)$ by using the operations $a + b = a \bigoplus b - (a \bigodot b)$, $a \cdot b = a \bigodot b$ in R. These transformations are inverses in that $b[r(B)] = B$ and $r[b(R)] = R$.

4 Cartesian Product

The Cartesian product is one of the most important constructions of set theory: it enables us to express many concepts in terms of sets.

Definition 4.1. With each two objects a, b, there corresponds a new object (a, b), called their ordered pair. Ordered pairs are subject to the one condition: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$; in particular, $(a, b) = (b, a)$ if and only if $a = b$. The first (second) element of an ordered pair is called its first (second) coordinate.

Remark: The concept of an ordered pair can be expressed in terms of sets by defining $(a, b) = \{\{a\}, \{a, b\}\}$; in your homework show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. For the sequel, we need only know that ordered pairs are uniquely determined by their first and second coordinates; the means for accomplishing this are immaterial.

Definition 4.2. Let A, B be two sets, distinct or not. Their cartesian product, $A \times B$, is the set of all ordered pairs $\{(a, b) | a \in A, b \in B\}$.

We have the basic

Proposition 4.1. $A \times B = \emptyset \Leftrightarrow [A = \emptyset] \vee [B = \emptyset]$.

The proof of this and the next statements we leave as a homework.

Proposition 4.2. If $C \times D \neq \emptyset$, then $C \times D \subset A \times B$ if and only is $[C \subset A] \wedge [D \subset B]$.

It follows at once from this and Proposition (2.3) that for nonempty sets A, B, $A \times B = B \times A$ if and only if $A = B$; the operation $A \times B$ is therefore not commutative.
The relation of $\times$ to $\bigcup$ and $\bigcap$ is summarized in the following trivial

Theorem 4.1. $\times$ *distributes over* $\bigcup$, $\bigcap$, *and* $-$:

$$A \times (B \bigcup C) = A \times B \bigcup A \times C,$$
$$A \times (B \bigcap C) = A \times B \bigcap A \times C$$
$$A \times (B - C) = A \times B - A \times C$$

The cartesian product of three sets A, B, C is defined by $A \times B \times C = (A \times B) \times C$, and that of n sets by induction: $A_1 \times \ldots \times A_n = (A_1 \times \ldots \times A_{n-1}) \times A_n$; an element of $A_1 \times \ldots \times A_n$ is written $(a_1, \ldots, a_n)$, and a_i is called ith coordinate.

5 Families of Sets

Definition 5.1. If to each element α of some set $\mathscr{A} \neq \emptyset$ there correspond a set A_α, then the collection of sets $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is called a *family* of sets, and $\mathscr{A}$ is called an indexing set for the family.

This definition does not required that sets with distinct indices be different. Observe that any set $\mathscr{F}$ of sets can be converted to a family of sets by "self-indexing": one uses the set $\mathscr{F}$ itself as an indexing set and assigns to each member of the set $\mathscr{F}$ the set it represents. In this section we extend the notions of union and intersection to families of sets; it should be noted that this is not done by any limiting process, but rather by independent definitions that reduce to the previous ones whenever family is finite.

Definition 5.2. Let Γ be a given set, and $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ a family of subsets of Γ. The union $\bigcup\limits_\alpha A_\alpha$ of this family is the set

$$\{x \in \Gamma \mid \exists \alpha \in \mathscr{A} : x \in A_\alpha\},$$

and the intersection $\bigcap\limits_\alpha A_\alpha$ is the set

$$\{x \in \Gamma \mid \forall \alpha \in \mathscr{A} : x \in A_\alpha\}.$$

We frequently denote $\bigcup\limits_\alpha A_\alpha$ by $\bigcup\limits_{\alpha \in \mathscr{A}} A_\alpha$ and by $\bigcup\{A_\alpha \mid \alpha \in \mathscr{A}\}$; similarly for $\bigcap\limits_\alpha A_\alpha$.

Theorem 5.1.

1. $\bigcup_\alpha$ *distributes over* $\cap$ *and* $\bigcap_\alpha$ *distributes over* $\cup$*:*

$$\left[\bigcup\{A_\alpha \mid \alpha \in \mathscr{A}\}\right] \cap \left[\bigcup\{B_\beta \mid \beta \in \mathscr{B}\}\right] = \bigcup\{A_\alpha \cap B_\beta \mid (\alpha,\beta) \in \mathscr{A} \times \mathscr{B}\}$$
$$\left[\bigcap\{A_\alpha \mid \alpha \in \mathscr{A}\}\right] \cup \left[\bigcap\{B_\beta \mid \beta \in \mathscr{B}\}\right] = \bigcap\{A_\alpha \cup B_\beta \mid (\alpha,\beta) \in \mathscr{A} \times \mathscr{B}\}$$

2. *If complements be taken with respect to* Γ*, then*

$$\mathbf{C}\Big(\bigcup_\alpha A_\alpha\Big) = \bigcap_\alpha \mathbf{C}A_\alpha \text{ and } \mathbf{C}\Big(\bigcap_\alpha A_\alpha\Big) = \bigcup_\alpha \mathbf{C}A_\alpha$$

3. $\bigcup_\alpha$ *and* $\bigcap_\alpha$ *distribute over the cartesian product:*

$$\bigcup\{A_\alpha \mid \alpha \in \mathscr{A}\} \times \bigcup\{B_\beta \mid \beta \in \mathscr{B}\} = \bigcup\{A_\alpha \times B_\beta \mid (\alpha,\beta) \in \mathscr{A} \times \mathscr{B}\}$$
$$\bigcap\{A_\alpha \mid \alpha \in \mathscr{A}\} \times \bigcap\{B_\beta \mid \beta \in \mathscr{B}\} = \bigcap\{A_\alpha \times B_\beta \mid (\alpha,\beta) \in \mathscr{A} \times \mathscr{B}\}$$

So far, we have assumed that $\mathscr{A} \neq \emptyset$; there are some formal advantages in allowing the indexing set to be the null set. If $\mathscr{A} = \emptyset$, Definition (5.2) gives that $\bigcup_\alpha\{A_\alpha \mid \alpha \in \mathscr{A}\} = \emptyset$, since no $x \in \Gamma$ satisfies the condition $\exists\, \alpha \in \mathscr{A} : x \in A_\alpha$; similarly we find that $\bigcap_\alpha\{A_\alpha \mid \alpha \in \mathscr{A}\} = \Gamma$, where Γ is the specified domain of discourse. In the future, we will call a family $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ *nonempty* whenever we wish to emphasize that the indexing set $\mathscr{A} \neq \emptyset$; this, of course, does not exclude the possibility that some $A_\alpha = \emptyset$.

6 Power Set

Definition 6.1. Let A be any set. Its power set $\mathscr{P}(A)$ is the set of all subsets of A.

Theorem 6.1. $\bigcap_\alpha$ *and* $\mathscr{P}$ *commute:* $\bigcap_\alpha \mathscr{P}(A_\alpha) = \mathscr{P}\Big(\bigcap_\alpha A_\alpha\Big)$*. Though* $\bigcup_\alpha$ *and* $\mathscr{P}$ *do not commute,* $\bigcup_\alpha \mathscr{P}(A_\alpha) \subset \mathscr{P}\Big(\bigcup_\alpha A_\alpha\Big)$*.*

7 Functions, or Maps

Definition 7.1. Let X and Y be two sets. A map $f : X \to Y$ (or function with *domain* X and *range* Y) is a subset $f \subset X \times Y$ with the property: for each $x \in X$, there is one, and only one, $y \in Y$ satisfying $(x, y) \in f$.

Definition 7.2. Let $f : X \to Y$. Then:

1 For each $A \subset X$, $f(A) = \{f(x) \mid x \in A\} \subset Y$ is called the image of A in Y under f.

2 For each $B \subset Y$, $f^{-1}(B) = \{x \mid f(x) \in B\} \subset X$ is called the inverse image of B in X under f.

Let $f : X \to Y$ be given. Then f *induces* a map (still denoted by f), $f : \mathscr{P}(X) \to \mathscr{P}(Y)$ by $A \to f(A)$. The map $f : X \to Y$ also induces a map $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$ by $B \to f^{-1}(B)$. Of these two maps, f^{-1} is the most important because of the following theorem:

Theorem 7.1. *Let $f : X \to Y$.Then the induced $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$ preserves the elementary set operations. Precisely,*

1. $f^{-1}\Big(\bigcup_\alpha B_\alpha\Big) = \bigcup_\alpha f^{-1}(B_\alpha)$.
2. $f^{-1}\Big(\bigcap_\alpha B_\alpha\Big) = \bigcap_\alpha f^{-1}(B_\alpha)$.
3. $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

In contrast to the (7.1) the induced map $f : \mathscr{P}(X) \to \mathscr{P}(Y)$ behaves less satisfactory. Though it preserves unions, it does not in general preserves intersections:

Proposition 7.1. If $f : X \to Y$, then for the induced map $f : \mathscr{P}(X) \to \mathscr{P}(Y)$:

1. $f\Big(\bigcup_\alpha A_\alpha\Big) = \bigcup_\alpha f(A_\alpha)$.
2. $f\Big(\bigcap_\alpha A_\alpha\Big) \subset \bigcap_\alpha f(A_\alpha)$.

For the combined action of f and f^{-1}, it is easy to verify

Proposition 7.2. If $f : X \to Y$, then:

1. For each $A \subset X$, $f^{-1}[f(A)] \supset A$.
2. For each $A \subset X$, and $B \subset Y$, $f[f^{-1}(B) \bigcap A] = B \bigcap f(X)$;
   in particular, $f[f^{-1}(B)] = B \bigcap f(X)$.

Given $f : X \to Y$ and $g : Y \to Z$, their composition $f \circ g : X \to Z$ is defined as the map $x \to g(f(x))$. We can clearly compose the induced maps f^{-1}, g^{-1} and we have

Theorem 7.2. *Let $f : X \to Y$ and $g : Y \to Z$. Then $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.*

Given an $f : X \to Y$ and a subset $A \subset X$, the map f considered *only* on A is called the *restriction* of f to A, is written $f \mid A$, and can alternatively be defined as $f \mid A = f \bigcap (A \times Y)$. In the reverse direction, if $A \subset X$ and $g : A \to Y$ is a given map, a map $G : X \to Y$ coinciding with g on A (that is, satisfying $G \mid A = g$) is called an *extension* of g over X relative to Y. The following result is very useful:

Theorem 7.3. *Let X be any set, and $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ any family of subsets with $\bigcup_\alpha A_\alpha = X$ (a "covering" of X). For each $\alpha \in \mathscr{A}$, let an $f_\alpha : A_\alpha \to Y$ be given, and assume that*

$$f_\alpha \mid A_\alpha \bigcap A_\beta = f_\beta \mid A_\alpha \bigcap A_\beta \text{ for each } (\alpha, \beta) \in \mathscr{A} \times \mathscr{A}.$$

Then there is one and only one, $f : X \to Y$ which is an extension of each f_α;
that is, $\forall \alpha \in \mathscr{A} : f \mid A_\alpha = f_\alpha$

If $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is covering of X, and if $A_\alpha \bigcap A_\beta = \emptyset$ whenever $\alpha \neq \beta$, then the family $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is called a *partition* of X. We obtain at once the

Corollary 7.1. *If $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is a partition of X and if for each $\alpha \in \mathscr{A}$ there is a given an $f_\alpha : A_\alpha \to Y$, then there exists a unique $f : X \to Y$ which is an extension of each f_α.*

If $f : X \to Y$ takes on every value in its range, f is called *surjective* (or a surjection; or "onto"). Observe that for surjective f (7.2)2 takes the simpler form: $\forall B : B \subset Y \Rightarrow f[f^{-1}(B)] = B$.

If f sends distinct elements of X to distinct elements of Y, f is called *injective* (or an injection; or one-to-one). Evidently f is injective if and only if $[x \neq x'] \Rightarrow [f(x) \neq f(x')]$, or equivalently, $[f(x) = f(x')] \Rightarrow [x = x']$. The restriction of an injection to any subset is also an injection.

If f is both injective and surjective, f is called *bijective* (or a bijection; or a one-to-one onto map). Clearly, $f : X \to Y$ is bijective if and only if $\forall y \in Y : f^{-1}\{y\}$ is a single point. Thus, with each bijection $f : X \to Y$, we have also a *map* $f^{-1} : Y \to X$ determined by $y \to f^{-1}(\{y\})$ (this differs from $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$, since the domains are not the same); it is evident that $f^{-1} : Y \to X$ is also bijective and that $(f^{-1})^{-1} = f$.

The following proposition indicates a simple method for establishing that a given map f (respectively g) is injective (respectively surjective).

Proposition 7.3. Let $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = 1_X$. then f is injective and g is surjective.

Proof: The map f is injective, since $[f(x) = f(x')] \Rightarrow x = g \circ f(x) = g \circ f(x') = x'$. g is surjective, since for any $x_0 \in X$, $x_0 = g[f(x_0)]$.

8 Binary Relations; Equivalence Relations

A binary relation R in a set A is, intuitively, a proposition such that for each ordered couple (a, b) of elements of A, one can determine whether aRb ("a is in relation R to b") is or is not true. We state this formally in terms of the set concept.

Definition 8.1. A binary relation R in a set A is a subset $R \subset A \times A$. $(a, b) \in R$ is written aRb.

Definition 8.2. A binary relation R in A is called an *equivalence* relation if:

1. $\forall a \in A : aRa$ (reflexive)
2. $(aRb) \Rightarrow (bRa)$ (symmetric)
3. $(aRb) \wedge (bRc) \Rightarrow (aRc)$ (transitive).

If aRb, we say that a and b are equivalent.

Let R be an equivalence relation in A. For each $a \in A$, the subset $Ra = b \in A \mid bRa$ is called the *equivalence class of a*. The fundamental theorem on equivalence relations is a consequence of

Lemma 8.1. *Let R be an equivalence relation in A. Then:*

1. $\bigcup\{Ra \mid a \in A\} = A$.
2. *If (aRb), then $Ra = Rb$.*
3. *If $\neg(aRb)$, then $Ra \cap Rb = \emptyset$.*

Theorem 8.1. *Let A have an equivalence relation R. Then the collection of distinct equivalence classes partitions A into mutually disjoint sets, called R-equivalence classes, such that any two elements of A belong to a common R-equivalence class if, and only if, they are equivalent.*

Each element a of an R-equivalence class A_a is called a *representative* of A_a; observe that $a \in A_a \Leftrightarrow A_a = Ra$.
With each equivalence relation R in A, we construct a new set according to the following

Definition 8.3. Let A have an equivalence relation R. The set whose *elements* are the R-equivalence classes is called the quotient set of A by R and is written A/R. The map $p_A : A \to A/R$ given by $a \to Ra$ is called the projection of A onto A/R.

Clearly, p_A is surjective, but not in general injective (since $Ra = Rb = A_a$ whenever aRb); note also that $A/R \subset \mathscr{P}(A)$. We omit the subscript on p_A when no confusion arises. A set of elements, one from each equivalence class, is called a *system of representatives* for A/R.

Let A, B be two sets with equivalence relations R, S respectively. A map $f : A \to B$ is called *relation-preserving* if $aRa' \Rightarrow f(a)Sf(a')$.

Theorem 8.2. *Let $f : A \to B$ be relation-preserving. Then there is one, and only one, map $f_* : A/R \to B/S$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p_A}\big\downarrow & & \big\downarrow{\scriptstyle p_B} \\ A/R & \xrightarrow[f_*]{} & B/S \end{array}$$

commutes (that is, $p_B \circ f = f_* \circ p_A$). f_* is called the map induced by f in "passing to the quotient". Conversely, if for any *two maps f, f_* the above diagram commutes, then f is necessarily relation-preserving, and f_* is the map induced by f.*

9 Axiomatics

Though the intuitive idea of calling any collection of objects a set will suffice for most purposes, an exposition of general Set Theory requires more precision, for without explicit axioms telling how the term *set* can be used, and to what collections it can be applied, various contradictions arise. There are several different axiomatic set theories, each having technical advantages and shortcomings; we present here a version based on the Bernays-Gödel-von Neumann axiomatics. The treatment is not intended to be either complete or formal, nor is the system of axioms asserted to be independent; these matters properly belong to the domain of Logic. It is desired to indicate only a framework within which we will work, which avoids the known antinomies and which, at least until now, has not led to any contradictions.

Ideally, we would like to have associated with each property p a set $E(p)$ consisting of all objects having property p. The assumption that this is true leads at once to the Russell antimony of the set of all sets not members of themselves: assuming that the property $p(x) = (x\, is\, a\, set) \wedge (x \neq x)$ determines some *set* $\mathscr{R}(p)$, we must conclude that $[\mathscr{R}(p) \notin \mathscr{R}(p)] \Leftrightarrow [\mathscr{R}(p) \in \mathscr{R}(p)]$. To block this contradiction, we adopt the attitude that it is not the conversion of properties to collections that is at fault, but rather the assumption that $\mathscr{R}(p)$ is a *set* and is therefore eligible for membership in the collection determined by p. The basic idea of this approach is, then, that there are two types of collections - classes and sets: any collection of objects specified by some property is a class, whereas only those classes that can be *members* of a class are *sets*. Heuristically, a set is a class that can be regarded as a single entity.

The undefined terms in the axiomatic development are "class" and a dyadic relation $\in$ between classes, the statement $\mathscr{A} \in \mathscr{B}$ is either true or false. A property p will mean a formula built up from statements "$\mathscr{A} \in \mathscr{B}$" by negation, conjunction, disjunction, and quantification of class variables by means of predicate calculus.

We begin by defining classes to be equal if they have the same members; formally,

Definition 9.1.

$$(\mathscr{A} \subset \mathscr{B}) \Leftrightarrow (\forall x : x \in \mathscr{A} \Rightarrow x \in \mathscr{B}),$$

and

$$(\mathscr{A} = \mathscr{B}) \Leftrightarrow (\mathscr{A} \subset \mathscr{B}) \wedge (\mathscr{B} \subset \mathscr{A})$$

This definition permits substitution with respect to the second class variable in the relation $x \in \mathscr{A}$; that is, $(x \in \mathscr{A}) \wedge (\mathscr{A} = \mathscr{B}) \Leftrightarrow (x \in \mathscr{B})$; to obtain it also for the first requires

Axiom 9.1. ***(of Individuality)*** $(x \in \mathscr{A}) \wedge (x = y) \Rightarrow (y \in \mathscr{A})$.

Next we distinguish between classes and sets by

Definition 9.2. The class A is called a set if there is a class $\mathscr{A}$ such that $A \in \mathscr{A}$.

Now we wish to postulate that any collection specified by a property that characterizes its members is a *class*. However, since nonsets cannot be members of anything, the members of a class must be *sets*. We formulate this by

Axiom 9.2. ***(of Class Formation)*** *For each property p in which only set variables are quantified and in which the class variable $\mathscr{A}$ does not appear, there is a class $\mathscr{A}$ whose members are just those* sets *having property p; in symbols,* $(x \in \mathscr{A}) \Leftrightarrow (x \textit{ is a set}) \wedge p(x)$.

Because of the Axiom (9.1), the class $\mathscr{A}$ is uniquely determined by its defining property; we will denote $\mathscr{A}$ by the notation $\{x \mid (x \textit{ is a set}) \wedge p(x)\}$ and sometimes by $\mathscr{A}(p)$. Observe that with this terminology, the Russell antinomy becomes the harmless statement

Proposition 9.1. The Russell class $\mathscr{R}(p)$ is not a set

Using Axiom (9.2), the Boolean operations $\mathscr{A} \bigcup \mathscr{B} = \{x \mid (x \in \mathscr{A}) \wedge (x \in \mathscr{B})\}$ and $\mathscr{A} \bigcap \mathscr{B} = \{x \mid (x \in \mathscr{A}) \vee (x \in \mathscr{B})\}$ with classes, as well as the cartesian product $\mathscr{A} \times \mathscr{B}$ of classes, are defined and are the classes. the universal class is $\{x \mid (x \textit{ is a set})\} \wedge (x = x)\}$, and the null class $\emptyset$ is $\{x \mid (x \textit{ is a set})\} \wedge (x \neq x)\}$; as in Proposition (2.4) , $\emptyset$ is unique and a subclass of every class. Equivalence relations in classes can be defined as in the previous section, leading to

Proposition 9.2. An equivalence relation in a class $\mathscr{A}$ partitions $\mathscr{A}$ into pairwise disjoint subclasses.

The next string of axioms guarantees at least one *set* and postulates that certain constructions using sets will yield sets.

Axiom 9.3. ***(of Null Set)*** *$\emptyset$ is a set.*

Axiom 9.4. ***(of Pairing)*** *If A, B are distinct sets, then $\mathscr{A} = \{x \mid (x = \mathscr{A}) \vee (x \in \mathscr{B})\}$ is a* set *(which contains exactly two elements). It is denoted by $\{A, B\}$.*

Axiom 9.5. ***(of Union)*** *If $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is a family of sets (recall that, as defined before, this means that $\mathscr{A}$ and each A_α are* sets*), then $\bigcup\{A_\alpha \mid \alpha \in \mathscr{A}\} = \{x \mid \exists \alpha \in \mathscr{A} : x \in A_\alpha\}$ is a set.*

Axiom 9.6. ***(of Replacement)*** *If A is a set and if $f :\rightarrow \mathscr{A}$ is a map, then $f(A)$ is a set.*

The next axiom deals with subset formation.

Axiom 9.7. ***(of Sifting)*** *If A is a set, then for any class $\mathscr{A}$, $A \bigcap \mathscr{A}$ is a set.*

In particular,

Proposition 9.3. If A is a set and p is a property in which only set variables are quantified, then $\{x \mid (x \in A) \wedge p(x)\}$ (which we will write as $\{x \in A \mid p(x)\}$) is a set.

For, if $\mathscr{A}$ is class determined by p, we have $A \bigcap \mathscr{A} = \{x \mid (x \in A) \wedge (x \textit{ is a set}) \wedge p(x)\}$, and the requirement $x \in A$ makes the stipulation "$(x$ is a set$)$" redundant.
Since members of classes must necessarily be sets, the precise definition of the power class $\mathscr{P}(\mathscr{A})$ of a class $\mathscr{A}$ is $\mathscr{P}(\mathscr{A}) = \{\mathscr{B} \mid (\mathscr{B} \textit{ is a set}) \wedge \mathscr{B} \subset \mathscr{A}\}$; thus, even if A is a set, $\mathscr{P}(\mathscr{A})$ has for members only those subclasses of A known to be sets.

Axiom 9.8. ***(of Power Set)*** *If A is a set, then $\mathscr{P}(\mathscr{A})$ is also a set.*

To indicate how these axioms are used, we establish that some frequently occurring constructions with sets will still yield sets.

Proposition 9.4. If $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is a family of sets, then $\bigcap\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is a set.

Proof. According to Axiom (9.5), $S = \bigcap\{A_\alpha \mid \alpha \in \mathscr{A}\}$ is a set; letting $p(x) = [\forall \alpha \in \mathscr{A} : x \in \mathscr{A}_\alpha]$, in which only the set variable α is quantified, Proposition (9.3) shows that $\{x \in S \mid p(x)\}$, which is precisely $S = \bigcap\{A_\alpha \mid \alpha \in \mathscr{A}\}$, is a set. □

Proposition 9.5. If A is a set, then so is $\{A\}$.

Proof. If $A = \emptyset$, then Axioms (9.3) and (9.8) give that $\{\emptyset\}$ is a set. If $A \neq \emptyset$, then $\{A, \emptyset\}$ is a set because of Axiom (9.4). Letting $\mathscr{A}$ be the class determined by the property $p(x) = (x = A)$, we find from Proposition (9.3) that $\{A, \emptyset\} \bigcap \mathscr{A} = \{A\}$ is a set. □

Proposition 9.6. The cartesian product of two sets is a set.

Proof. Let A, B be sets, and for each $a_0 \in A$ define a map $B \rightarrow A \times B$ by $b \rightarrow (a_0, b)$; according to Axiom (9.6), the image, $\{a_0\} \times B$, is a set; since $A \times B = \bigcup\{\{a_0\} \times B \mid a_0 \in A\}$, Axiom (9.5) shows that $A \times B$ is a set. □

Proposition 9.7. If A and B are sets, then the class of all maps $A \to B$ is a set.

Proof. We have just seen that $A \times B$ is a set so, by Axiom (9.6), $\mathscr{P}(A \times B)$ is also a set. Since a map is a subclass of $A \times B$ specified by some property, Proposition (9.3) shows that each map is a member of the set $\mathscr{P}(A \times B)$. Using now the property m expressed in Definition (7.1), the class of all maps $A \to B$ is $\{x \in \mathscr{P}(A \times B) \mid x \textit{ has property } m\}$ so, again by Proposition (9.6), it is a set. □

Proposition 9.8. The class of all sets is not a set.

Proof. Let $\mathscr{R}(p)$ be the Russell class. If the class A of all sets were a set, then Axiom (9.7) would imply that $A \bigcap \mathscr{R}(p)$ is a set; since $A \bigcap \mathscr{R}(p) = \{x \mid (x \textit{ is a set}) \wedge (x \neq x)\} = \mathscr{R}(p)$, this contradicts Proposition (9.1). □

We now add

Axiom 9.9. ***(of Foundation)*** *In each nonempty set A there is a $u \in A$ such that $u \cap A = \emptyset$ (that is, $\forall x : x \in A \Rightarrow \neg(x \in u)$).*

Loosely speaking, this axiom asserts that each nonempty set must contain "atoms" u, which form its "foundation". Its use is shown in

Proposition 9.9.

1. No nonempty set can be a member of itself.
2. If A, B are nonempty sets, then it is not possible that both $A \in B$ and $B \in A$ are true.

Proof.

1. Assume there were a nonempty set A such that $A \in A$; by Proposition (9.5), $\{A\}$ would also be a set, and because A is also the only member of A, the set A would not have a foundation.
2. Consider the set $\{A, B\}$ (using Axiom (9.4)) in an analogous way.

□

We now provide for the existence of infinite sets by

Axiom 9.10. ***(of Infinity)*** *There exists a* set *A with the properties: (i) $\emptyset \in A$, and (ii) if $a \in A$; then, $a \cup \{a\} \in A$.*

As an application we have

Proposition 9.10. The class of nonnegative integers is a set

Proof. Let A be any set having the two properties listed in Axiom (9.10)), and let $\mathscr{B} \subset \mathscr{P}(A)$ be defined by $\mathscr{B} = \{B \in \mathscr{P}(A) \mid B$ *has two properties in Axiom (9.10)*$\}$. Each B is a set, and by Proposition (9.3) and Axiom (9.8), so also is $\mathscr{B}$; it therefore follows from Proposition (9.4) that $N = \bigcap\{B \mid B \in \mathscr{B}\}$ is also a set. Because each B has the properties (i) and (ii) of Axiom (9.10), it is evident that N has them also. Refereeing now to the Peano axioms for the integers, and calling $x \cup \{x\} \in N$ the successor of $x \in N$, it can be easily verified that all the Peano axioms are satisfied by N [the principle of mathematical induction is valid because, by definition of N, N has no proper subset that satisfies both (i) and (ii)]. Since the Peano axioms are categorical, it follows that N can be regarded as the set of nonnegative integers. We denote $\emptyset$ by "0", $\{\emptyset\}$ by "1", $\{\emptyset, \{\emptyset\}\}$ by "2", and so on. $\square$

An easy consequence from this, Proposition (9.6) and Axiom (9.6), is that the class Q of rationals is a set; we will see later that there is a bijection of $\mathscr{P}(N)$ onto the reals; therefore by Axioms (9.6) and (9.8), Euclidian line E^1 is also a set.

According to this axiomatization, the only general method for producing subsets of a given set is that given in Proposition (9.3). To see that there are subsets that we would like to consider but that cannot be described by any property, consider the following example of Russell: Let A be an infinite collection of pairs of shoes; we can define a subset consisting of exactly *one* shoe from each pair by the property "right shoe". If now A is an infinite collection of pair of stocking, analogy would indicate that a subset consisting of exactly *one* stocking from each pair could be formed; but because stockings are identical in all respects, there can be no property that characterizes exactly one of each pair; in particular, we are not allowed to call such a collection a *subset* of A. Analogous situations arise frequently in mathematics; to give the broadest scope to mathematical considerations, we adopt as another method for producing subsets,

Axiom 9.11. ***(of Choice)*** *Given any nonempty family $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ of nonempty pairwise disjoint* sets, *there exists a* set *S consisting of exactly* one *element from each A_α.*

This is the only *existential* axiom: in contrast to all the others, a set obtained by application of this axiom is *not*, in general, uniquely determined by the given conditions. It has been shown (1938) by K. Gödel that if the set theory based on the first ten axioms is consistent, then the set theory based on all the eleven axioms is also consistent. Gödel's result obviously leaves open the possibility that Axiom (9.11) is derivable from the other axioms, and, in 1963, P.J. Cohen proved that it is not. Thus, the axiom of choice in fact an independent axiom.
Remark Note that appeal to Axiom (9.11) is not necessary if $\mathscr{A}$ if a *finite* set. Indeed, if $A_1, \cdots, A_n$ are the sets, then each A_i contains some element a_i so, by taking $p_i(x)$ to be property $(x = a_i)$, we obtain $\{x \in \bigcup_1^n \mid p_1(x) \vee \cdots \vee p_n(x)\}$ as a set satisfying the requirements of Axiom (9.11). However, if $\mathscr{A}$ is *infinite* (even though, as in Russell's example, each A_α is finite), some principle such as Axiom (9.11) appears to be necessary. A property such as "contains one element from

each A_α"is illegitimate because properties carve out *unique* subsets and it is evident that if there is one collection satisfying the proposed predicate, then there are others also (unless A_α consists of a single element). taking $p_i(x) = (x = a_i)$, the procedure used above for finite $\mathscr{A}$ cannot be emulated, since an infinite "or" chain is *not* a proposition. And to use a predicate such as $\exists\, i : p_i(x)$ is inadequate: for, assuming about x that $\neg\, p_1(x),\ \neg\, p_2(x),\ \cdots$, one cannot in general conclude $\forall\, i :\ \neg p_i(x)$, that is, $\neg\exists i :\ p_i(x)$ [or, that x does not belong to the set determined by $\exists\, i :\ p_i(x)$], without tampering with the rules of logic in the predicated calculus; from this viewpoint, the need for Axiom (9.11) is related to the ω-incompleteness of consistent systems.