MA651 Topology. Lecture 10. Metric Spaces.

This text is based on the following books:

- *"Topology" by James Dugundgji*
- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Linear Algebra and Analysis" by Marc Zamansky*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

57 Metrics on sets

Definition 57.1. A metric (or distance function) on a set Y is a map: $d : Y \times Y \to E^1$ with the properties:

1. $d(x, y) \geqslant 0$ for each pair x, y
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ for all x, y (symmetry)
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ for each triple of points (triangle inequality)

$d(x, y)$ is called the distance between x and y.

The absence of the axiom (3) is easily remedied:

$$d'(x, y) = d(x, y) + d(y, x)$$

gives a metric d' in Y itself.

Example 57.1. *Every non-empty set Y can be considered as a metric space. We can define d by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.*

Example 57.2. *In the set of all real numbers* E^1, $d(x, y) = |x - y|$ *is a metric. More generally, in the set* E^n, $d_0 = max\{|x_i - y_i| \mid 1 \leqslant i \leqslant n\}$ *is a metric: the triangle inequality follows because*

$$|x_i - z_i| \leqslant |x_i - y_i| + |y_i - z_i| \leqslant d_0(x, y) + d_0(y, z)$$

for each $1 \leqslant i \leqslant n$

Example 57.3. *In the set* E^n,

$$d_p(x, y) = \sqrt[p]{\sum_1^n |x_i - y_i|^p}$$

is a metric space for $p \geqslant 1$. *To verify the triangle inequality, note that for* $p \geqslant 1$, *the function* $f(x) = x^p$ *satisfies* $f''(x) \geqslant 0$ *on* $\{x \in E^1 \mid x \geqslant 0\}$ *and so is convex, that is, it satisfies* $f(\lambda x + [1 - \lambda]y) \leqslant \lambda f(x) + [1 - \lambda]f(y)$ *for* $x, y \geqslant 0$. *Thus, for any set of real numbers* $a_1, \cdots, a_n$, $b_1, \cdots, b_n$, *letting*

$$A = \sqrt[p]{\sum_1^n |a_i|^p}, \quad B = \sqrt[p]{\sum_1^n |b_i|^p}$$

and expressing the convexity of x^p *for each* $1 \leqslant i \leqslant n$ *with*

$$x = \frac{a_i}{A}, \quad y = \frac{b_i}{B}, \quad \lambda = \frac{A}{A + B},$$

addition of these n *inequalities gives Minkowski's inequality*

$$\sqrt[p]{\sum_1^n |a_i + b_i|^p} \leqslant \sqrt[p]{\sum_1^n |a_i|^p} + \sqrt[p]{\sum_1^n |b_i|^p}$$

the triangle inequality follows by taking $a_i = x_i - y_i$, $b_i = y_i - z_i$, *for* $1 \leqslant i \leqslant n$.

Example 57.4. *The set* $C_{[a,b]}$ *of all continuous functions defined on the closed interval* $[a, b]$, *with distance*

$$\rho(f, g) = \max_{a \leqslant t \leqslant b} |f(t) - g(t)|$$

is a metric space with great importance in analysis.

Example 57.5. *More generally than in Example (57.4). Let* Y *be any set and* $C_Y = \{f \mid f : Y \to E^1$ *and* f *is bounded*$\}$. *Then* $d(f, g) = sup\{|f(y) - g(y)| \mid y \in Y\}$ *is a metric in* $C(Y)$. *Note that if some* f_0 *were unbounded,* d *would not be a metric: using* $g(y) \equiv 0$, *the function* d *is not defined at the point* $(f_0, g) \in C(Y) \times C(Y)$

58 Topology induced by a metric

With each metric d in a set Y, we are going to associate a definite topology $\mathscr{T}(d)$ in Y.
The set $B_d(a,r) = \{y \mid d(y,a) < r\}$ is called the $d-$ball of radius r and center a. Clearly, $B_d(a,r) \subset B_d(a,r')$ if $r \leqslant r'$ and $B_d(a,0) = \emptyset$. In the future, we will omit the distinguishing d whenever the metric is clear from context.

Proposition 58.1. The family $\{B_d(y,r) \mid y \in Y, r > 0\}$ of all $d-$balls in Y can serve as the basis for a topology.

Proof. We need to verify that if

$$a \in B_d(x_0, r_x) \cap B(y_0, r_y),$$

then a belongs to some ball lying in this intersection. Let

$$r = \min[r_x - d(a,x_0), r_y - d(a,y_0)]$$

then $r > 0$, since the statements "$a \in B(\xi, r_\xi)$" and "$d(a,\xi) < r_\xi$" are equivalent so $a \in B(a,r)$. Furthermore, $B(a,r) \subset B_d(x_0,r_x) \cap B(y_0,r_y)$ because if $x \in B(a,r)$, then $d(x,x_0) \leqslant d(x,a) + d(a,x_0) < r + d(a,x_0) \leqslant [r_x - d(a,x_0)] + d(a,x_0) = r_x$; that is, $x \in B(x_0,r_x)$ and, similarly, $x \in B(y_0,r_y)$. $\square$

Definition 58.1. Let Y be a set and d be a metric in Y. The topology $\mathscr{T}(d)$, having for basis the family $\{B_d(y,r) \mid y \in Y, r > 0\}$ of all $d-$balls in Y, is called the topology in Y induced by the metric d.

The converse question arises: Given a topological space $(X, \mathscr{T})$, is there a metric d in X such that $\mathscr{T} = \mathscr{T}(d)$? The answer is "no" in general. For example, no metric can induce the topology in Sierpinski space.

Definition 58.2. A topological space $(X, \mathscr{T})$ is called a metric (or metrizable) space if its topology is that induced by a metric in Y. A metric for a space Y is one that induces its topology.

With this terminology, the Euclidean space E^n is a metric space, and d_0 is a metric for this space. In metric spaces, topological concepts can be phrased in the ε, δ terms of classical analysis. For example:

Example 58.1. *Let X have topology $\mathscr{T}(d)$ and Y have topology $\mathscr{T}(\rho)$. An $f : X \to Y$ is continuous if*

$$\forall x \,\forall \varepsilon > 0 \,\exists\, d(\delta, x) < \delta \Rightarrow \rho(f(\xi), f(x)) < \varepsilon;$$

that is, if $f[B_d(x,\delta)] \subset B_\rho[f(x), \varepsilon]$.

59 Equivalent metrics. Isometries.

In this section we give a criterion for determining in advance whether two different metrics in a set Y will induce the same topology.

Definition 59.1. Two metrics d, ρ, in a set Y are called equivalent, $d \sim \rho$, if $\mathscr{T}(d) = \mathscr{T}(\rho)$.

This is clearly an equivalence relation in the set of all metrics on Y.

Theorem 59.1. *Let d, ρ be two metrics on Y. A necessary and sufficient condition that $d \sim \rho$ is that for each $a \in Y$ and $\varepsilon > 0$, the following two conditions hold:*

1. $\exists\, \delta_1 = \delta_1(a, \varepsilon) : \rho(a, y) < \delta_1 \Rightarrow d(a, y) < \varepsilon.$
2. $\exists\, \delta_2 = \delta_2(a, \varepsilon) : d(a, y) < \delta_2 \Rightarrow \rho(a, y) < \varepsilon.$

Proof is left as a homework.

Example 59.1. *In the set E^n the following metrics are equivalent:*

1. $d_{max}(x, y) = max\{|x_i - y_i| \mid 1 \leqslant i \leqslant n\}$
2. $d_p(x, y) = \sqrt[p]{\sum\limits_1^n |x_i - y_i|^p}$
3. $d(x, y) = \sum\limits_1^n |x_i - y_i|$

Proof is left as a homework.

Corollary 59.1. *Let $(Y, \mathscr{T}(d))$ be a metric space. Then for each $M > 0$ there is a metric $\rho_M \sim d$ such that $\rho_M(x, y) \leqslant M$ for all (x, y). Equivalently, each metric space is homeomorphic to a bounded metric space.*

Proof. Given M, define $\rho_M(x, y) = \min[M, d(x, y)]$; it is trivial to verify that ρ_M is indeed a metric for Y and, using Theorem (59.1), that $d \sim \rho_M$. □

Given metric spaces (X, ρ) and (Y, d), a map $f : X \to Y$ is said to be distance-preserving if $d(f(x), f(y)) = \rho(x, y)$ whenever $x, y \in X$. A distance preserving bijection is called an isometry.

Definition 59.2. f if a (ρ, d) isometry if and only if $f : X \to Y$ is a bijection and $\rho(x, y) = d(f(x), f(y))$ whenever $x, y \in X$.

It is easy to see that an isometry is an homeomorphism. Of course, not every homeomorphism between metric spaces is an isometry. For example, the map $f : E^1 \to E^1$ given by $f(x) = \frac{1}{2}x$ shrinks distances, but is a homeomorphism.

Theorem 59.2. *Let ρ, d be metric on X, Y respectively. Let $f : X \to Y$ be a (ρ, d) isometry. Then $f^{-1} : Y \to X$ is a (d, ρ) isometry.*

Proof is left as a homework.

60 Continuity of distance.

Definition 60.1. In a metric space Y, with metric d,

1. The distance of a point y_0 to a nonempty set A is

$$d(y_0, A) = \inf\{d(y_0, a) \mid a \in A\}$$

2. The distance between nonempty sets A and B is

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\} = \inf\{d(a, B) \mid a \in A\}.$$

3. The diameter of a nonempty set A is

$$\delta(A) = \sup\{d(x, y) \mid x \in A, y \in A\}$$

By convention, $\delta(\emptyset) = 0$. A set A is called bounded if $\delta(A) < \infty$; d is a bounded metric is $\delta(Y) < \infty$.

Example 60.1. *$d(A, B) \neq 0 \Rightarrow A \cap B = \emptyset$; the converse implication need not be true, even though both A and B are closed: In E^1, let*

$$A = \{n \mid n \in Z^+\} \text{ and } B = \{n + (1/2n) \mid n \in Z^+\}.$$

To obtain this converse implication at least in the trivial cases that $A = \emptyset$ or $B = \emptyset$, we define $d(A, \emptyset) = d(\emptyset, B) = \infty$.

Proposition 60.1.

(a) $d(y, A) = 0$ if and only if $y \in \overline{A}$; thus $\overline{A} = \{y \mid d(y, A) = 0\}$

(b) $\rho \sim d$ if and only if for each $A \subset Y$, $d(y, A) = 0 \Leftrightarrow \rho(y, A) = 0$

Proof.

(a) We have $y \in \overline{A} \Leftrightarrow \exists B(y, r) : A \cap B(y, r) \neq \emptyset \Leftrightarrow \forall\, r > 0\, \exists\, a_r \in A : d(y, a_r) < r \Leftrightarrow d(y, A) = 0$

(b) By (a) $\mathscr{T}(d)$ and $\mathscr{T}(\rho)$ give the same closure operation in Y, so $\mathscr{T}(d) = \mathscr{T}(\rho)$

□

Theorem 60.1. *Let Y be a metric space, with metric d, and let A be any subset of Y. Then the mat $f : Y \to E^1$ defined by $y \to d(y, A)$ is continuous.*

Proof. Let x, y be any two elements of Y. Then, for each $a \in A$, we have $d(x,a) \leqslant d(x,y)+d(y,a)$ so that

$$d(x,A) = \inf_{\alpha} d(x,a) \leqslant d(x,y) + \inf_{\alpha} d(y,a) = d(x,y) + d(y,A),$$

which shows that $d(x,A) - d(y,A) \leqslant d(x,y)$. Interchanging the roles of x and y, we obtain

$$|d(x,A) - d(y,A)| \leqslant d(x,y)$$

which gives the continuity of f. □

61 Convergent sequences

In many questions concerning the metric spaces we may use convergent sequences in place of more general topological concepts.

First we observe that a sequence (x_n) in a metric space (X,d) converges to a point $x \in X$ if and only if $d(x,x_n)$ tends to zero.

Thus if x_n tends to x, every open ball of center x and radius ε contains all the x_n except for a finite number, and so for $\varepsilon > 0$ we have $d(x,x_n) < \varepsilon$ except for a finite set of values of n.

Conversely, if $d(x,x_n)$ tends to zero, for each $\varepsilon > 0$ the open ball of center x and radius ε contains all the x_n such that $d(x,x_n) < \varepsilon$, and so contains all but a finite number of the x_n.

Proposition 61.1. In a metric space (X,d) a point x is adherent to a set A if and only if there is a sequence (x_n) of points A which converges to x.

Proof. If $x_n \in A$ and x_n tends to x every open ball of center x contains an x_n and so meets A. Conversely, if $x \in \overline{A}$ every ball of center x and radius $1/n$ meets A and so contains a point of A

which we shall call x_n. The sequence (x_n) converges to x since $d(x,x_n) < 1/n$. □

Proposition 61.2. x is a limit point of a sequence (x_n) if and only if there is a subsequence of (x_n) which converges to x.

Proof. If x is a limit point of (x_n) there exists x_{n_1} such that $d(x,x_{n_1}) < 1$. We can then find $n_2 > n_1$ such that $d(x,x_{n_2}) < 1/2$, etc. In this way we construct a sequence (x_{n_k}) which converges to x.

Conversely, if x is the limit of a subsequence (x_{n_k}) of (x_n), every open ball of center x contains all but a finite number of the (x_{n_k}) and so meets every set containing all but finite number of the (x_n).

□

Proposition 61.3. A mapping f of a metric space (E, d) into a topological space F is continuous at a point $x_0 \in E$ if and only if for every sequence (x_n) of points of E converging to x_0, $f(x_n)$ converges to $f(x_0)$ in F.

Proof. Suppose f is continuous at the point x_0 and let $y_0 = f(x_0)$. let (x_n) be a sequence converging to x_0 in E. Then for every open neighborhood Y of y_0 in F there is an open ball $B(x_0, \rho)$ of center x_0 such that for every $x \in B(x_0, \rho)$ we have $f(x) \in Y$. Since x_n converges to x_0, $x_n \in B(x_0, \rho)$ and so $f(x_n) \in Y$ except for a finite set of values of n.

Conversely, suppose that for every sequence (x_n) converging to x_0 in E, $f(x_n)$ converges to $y_0 = f(x_0)$ in F, and that f is not continuous at x_0. Then there is an open set Y containing y_0 such that $f^{-1}(Y)$ contains no open ball of E which contains x_0. Let $\rho_1 > 0$, then in the open ball $B(x_0, \rho_1)$ there is a point x_1 such that $f(x_1) \notin Y$. Since $f(x_0) = y_0 \in Y$, x_1 is not equal to x_0. Now let $\rho_2 > 0$ be less than both $\rho_1/2$ and $(1/2)d(x_0, x_1)$. $B(x_0, \rho_2)$ contains $x_2 \neq x_0$ such that $f(x_2) \notin Y$. In this way we construct a sequence (x_n) of points of E converging to x_0, since $d(x_0, x_n) \leqslant \rho_1/2^n$, and such that for every n, $f(x_n) \notin Y$. this last property contradicts the hypothesis that $f(x_n)$ converges to y_0, for Y must contain all the $f(x_n)$ except possibly those corresponding to a finite set of values of n. $\square$

Example 61.1. *The numerical function $(x, y) \to d(x, y)$ defined on $X \times X$ is continuous there. If*

suffices to show that is (x_n, y_n) tends to (x_0, y_0) for the distance defined on $E \times E$, then $d(x_n, y_n)$ tends to $d(x_0, y_0)$. Now the distance of (x_n, y_n) from (x_0, y_0) in $E \times E$ is $d(x_n, x_0) + d(y_n, y_0)$, so that the inequality

$$|d(x_n, y_n) + d(x_0, y_0)| \leqslant d(x_n, x_0) + d(y_n, y_0)$$

established the result.

62 Compactness and covering theorems in metric spaces

Theorem 62.1. *Every metric space is $1°$-countable.*

Proof. Let $(Y, \mathscr{T}(d))$ be a metric space. By the definition of the topology $\mathscr{T}(d)$, it is evident that $\{B_d(y, r) \mid r \text{ rational}\}$ is a countable basis at y. $\square$

The notions of countable compactness and sequential compactness are equivalent in metric spaces. We would like to improve upon this result and conclude that these are all in fact equivalent to compactness in metric spaces.

The proof of this is not so easy. As a means of attack, we shall first show that each countably compact metric space is $2°$-countable. This in itself requires an involved argument, so we precede it with a simplifying Lemma, which states that each countably compact space can be covered by finitely many spheres of arbitrary small radius. Such a space is often called totally bounded in the literature.

Definition 62.1. $(X, \mathscr{T}\rho)$ is totally bounded if and only if for $\varepsilon > 0$, there is some $n \in Z^+$ and there are points $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_\rho(x_i, \varepsilon)$.

Lemma 62.1. *Let $(X, \mathscr{T}(\rho))$ be a countably compact metric space, then X is totally bounded.*

Theorem 62.2. *Let $(X, \mathscr{T}(\rho))$ be a countably compact metric space, then X is 2°-countable.*

Theorem 62.3. *$(X, \mathscr{T}(\rho))$ is countably compact $\Leftrightarrow$ $(X, \mathscr{T}(\rho))$ is compact.*

Also if the space is metric the converse for the Theorem (56.4) (Every 2°-countable space is Lindelöf.) is true:

Theorem 62.4. *Let $(X, \mathscr{T}(\rho))$ be a metric space, then*

$$X \text{ is Lindelöf} \Leftrightarrow X \text{ is } 2^\circ\text{-countable}$$

At the same time

Theorem 62.5. *Let $(X, \mathscr{T}(\rho))$ be a metric space, then*

$$X \text{ is separable} \Leftrightarrow X \text{ is } 2^\circ\text{-countable}$$

The summary for the metric space properties is

Theorem 62.6. *For the metric spaces we have the following implications:*

1. *$(X, \mathscr{T}(\rho))$ is compact $\Leftrightarrow$ countably compact $\Leftrightarrow$ sequentially compact*
2. *$(X, \mathscr{T}(\rho))$ is 2°-countable $\Leftrightarrow$ separable $\Leftrightarrow$ Lindelöf*
3. *$(X, \mathscr{T}(\rho))$ is 1°-countable*

And $(1) \Rightarrow (2) \Rightarrow (3)$