MA651 Topology. Lecture 11. Metric Spaces 2.

This text is based on the following books:

- *"Topology" by James Dugundgji*
- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Linear Algebra and Analysis" by Marc Zamansky*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

63 Complete metric spaces

63.1 Cauchy sequences

The importance of complete metric spaces lies in the fact that in such spaces we can decide whether a sequence is convergent, without necessarily knowing its limit.

Definition 63.1. Let (E, d) be a metric space. A sequence (x_n) in E is called a Cauchy sequence (d-Cauchy sequence) if

$$\lim_{p,q\to\infty} d(x_p, x_q) = 0$$

We recall that

$$\lim_{p,q\to\infty} d(x_p, x_q) = 0$$

means:

$$\forall\, \varepsilon > 0\ \exists N(\varepsilon)\ \forall p, q \geqslant N\ :\ d(x_p, x_q) < \varepsilon$$

The notion of a d-Cauchy sequence is depended on the particular metric used: The same sequence can be Cauchy for one metric, but not Cauchy for an equivalent metric.

Example 63.1. *In E^1, the Euclidean metric $d_e(x, y) = |x - y|$ is equivalent to the metric*

$$d_\varphi(x, y) = |\frac{x}{1 + |x|} - \frac{y}{1 + |y|}|$$

since the latter is derived from the homeomorphism $x \to x/(1 + |x|)$ of E^1 and $]-1, +1[$. The sequence $\{n \mid n = 1, 2, \ldots\}$ in E^1 is not d_e-Cauchy sequence, whereas it is d_φ-Cauchy sequence.

Properties of Cauchy sequences

Properties of

1. *Every convergent sequence is Cauchy sequence.* For (x_n) is a convergent sequence in a metric space E then there exists $x \in E$ such that $d(x_n, x)$ tends to zero. The property then follows from

$$d(x_p, x_q) \leqslant d(x_p, x) + d(x, x_q)$$

   This property shows that the concept of a Cauchy sequence is more general than that of a convergent sequence, so the converse is false. In fact it is strictly more general, for example, there are metric spaces in which a Cauchy sequence may fail to converge, for example Q.

   Example 63.2. *In the space $Y =]0, 1]$ with the Euclidean metric d_e, the sequence $\{1/n\}$ is d_e-Cauchy, yet it does not converge to any $y_0 \in Y$.*

2. *If (x_n) is a Cauchy sequence and if for the sequence (y_n) we have $\lim d(x_n, y_n) = 0$, Then y_n is a Cauchy sequence.* This is so since

$$d(y_p, y_q) \leqslant d(y_p, x_p) + d(x_p, x_q) + d(x_q, y_q)$$

3. *If (x_n) is a Cauchy sequence so is every one of its subsequences* This is evident.

4. *If (x_n) is a Cauchy sequence containing a convergent subsequence (x_{n_k}) then (x_n) is convergent.*

   If, for a subsequence (x_{n_k}), there exists $x \in E$ such that

$$\lim_{k \to \infty} d(x_{n_k}, x) = 0$$

   then we have $d(x_{n_k}, x) < \varepsilon$ for $k \geqslant k_0$. But $d(x_p, x_q) < \varepsilon$ for $p, q \geqslant N$, so that then

$$d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x)$$

   and for $n \geqslant \max(N, n_0)$ (and $n_k \geqslant \max(N, n_{k_0})$) we have

$$d(x_n, x) \leqslant \varepsilon + \varepsilon$$

63.2 Complete metrics and complete spaces

Definition 63.2. Let Y be a metrizable space. A metric d for Y (that is, one that metrics the given topology of Y) is called complete if every d-Cauchy sequence in Y converges.

It must be emphasized that completeness is a property of metrics: One metric for Y may be complete, whereas another metric may not.

A given metric space Y may not be have any complete metric; to denote those that do, we have:

Definition 63.3. A metric space Y is called topologically complete (or briefly, complete) if a complete metric for Y exists. To indicate that d is a complete metric for Y, we say that Y is d-complete.

Less rigorously we can say: A metric space is complete if every Cauchy sequence in it is convergent. We can also say that a metric space is complete if the Cauchy sequences and convergent sequences are the same, or a sequence is convergent if and only if it is a Cauchy sequence.

The importance of this concept lies in the fact that if we have somehow or other verified that a space is complete it is no longer necessary to find the limit of a sequence in order to know that it is convergent.
In the other words, if a metric space is complete and if we have shown for a sequence (x_n), that $\lim d(x_p, x_q) = 0$, then we can assert that there is a point X of the space (and the only one) such that $\lim d(x_n, x) = 0$.

Theorem 63.1. *Let (Y, d) be a metric space, and assume that d has the property: $\exists\ \varepsilon > 0\ \forall y \in Y : \overline{B_d(y, \varepsilon)}$ is compact. Then d is complete.*

Proof. Let φ be a d-Cauchy sequence in Y, and choose n so large that $\delta[\varphi(T_n)] < \varepsilon/2$; then $\varphi(T_n) \subset \overline{B_d[\varphi(n), \varepsilon]}$ and therefore φ has an accumulation point y_0, thus $\varphi \to y_0$. $\square$

Corollary 63.1. *Every local compact metric space Y is topologically complete. Furthermore, if Y is compact, then every metric d for Y is complete.*

Proof is left as a homework.

Theorem 63.2. *A subset A of a complete metric space is a complete subspace if and only if it is closed.*

Proof. Let (Y, d) be a complete metric space and A a closed subset in Y. Let (x_n) be a Cauchy sequence in A for the induced distance. It converges in Y to a point x, but since A is closed, $x \in A$.

Conversely, let (Y, d) be a metric space and A a complete subspace. Let x be adherent to A so that it is the limit of a sequence (x_n) of elements of A. Since (x_n) converges to x in Y it is a

Cauchy sequence in Y and so also in A. Since A is complete the sequence converges to a point of A. Since the limit is unique it must be a point x. thus every adherent point belongs to A, and so A is closed.

$\square$

Remarks:

1. In the proof of the converse of this result we have shown that every complete subspace of a (not necessarily complete) metric space is closed.
2. This proposition shows that in a complete metric space the closed sets and the complete subspaces coincide (and the closed sets and the compact subspaces coincide in a compact space).

64 The Baire property of complete metric spaces

Theorem (63.1) gives a sufficient condition for topological completeness; the following theorem is a necessary condition. Because completeness is more prevalent than local compactness, this result is one of the most important and useful in topology, and has extensive applications in analysis.

Theorem 64.1. *In a complete metric space E a countable union of closed sets without interior points also has no interior points. (or, a countable intersection of open sets dense in E is also dense in E.)*

Definition 64.1. A topological space X is a Baire space if the intersection of each countably family of open sets in Y is dense.

Then the Theorem can be written as: Any topologically complete space is a Baire space.

Proof. Let (O_n) be a *countable* family of dense open sets and let $A = \bigcap O_n$. To show that $\bigcap O_n$ is everywhere dense it suffices to prove that for every non-empty open ball B in the space E we have

$$B \cap \bigcap O_n \neq \emptyset$$

Let (r_n) be a sequence of strictly positive numbers converging to 0.

If B is a non-empty open ball and O_1 an every where dense open set, $B \cap O_1$ is non-empty and open, and so contains a non-empty open ball.

Choose this ball B_1 to have radius less than r_1.

Since B_1 is non-empty and open, $B_1 \cap O_2$ is also non-empty and open and contains a non-empty open ball B_2 whose radius we may suppose to be less than r_2. In this way we construct, step by step, a countable family of non-empty open balls B_n with radii less than r_n, respectively, and such that

$$B_n \subset B_{n-1} \cap O_{n-1}$$

which implies that $B_n \subset B_{n-1}$ and $\bigcap B_n \subset B \cap \bigcap O_n$. It remains to prove that the B_n have a non-empty intersection.

Let x_n denote the center of B_n. For integers p such that $0 \leqslant p \leqslant q$, $x_q \in B_q$ so that $d(x_p, x_q) < r_p$. Since $\lim r_p = 0$, $d(x_p, x_q)$ tends to 0, and so the sequence (x_n) is a *Cauchy sequence.* Since E is *complete* (x_n) converges to a point $x \in E$. It follows that for an arbitrary integer p, and q tending to infinity:

$$d(x_p, x) \leqslant d(x_p, x_q) + d(x_q, x) < r_p + d(x_q, x)$$

and so since

$$\lim_{q \to \infty} d(x_q, x) = 0$$

it follows that $d(x_p, x) \leqslant r_p$ for all p and so $x \in B_p$ for all p, or $x \in \bigcap B_p$. The intersection of the balls B_n is thus non-empty and so

$$B \cap \bigcap O_n \neq \emptyset$$

□

Remark

Let O be a non-empty open set in a complete metric space E. The subspace O is not, in general, a complete metric space. But it is clearly has the Baire property for if A is open and dense in the subspace O it is open in E and $A \cup \mathbf{C}\overline{A}$ is an open set dense in E.

65 Completion of a metric space

The theorem on the completion of a metric space is important, because of its generality and because of its applications. It shows that every metric space can be embedded in a complete metric space. The situation is the same as for the set Q of rationals, which is not complete, but which can be embedded in the set R of real numbers, a complete space. The construction of the completion of a matric space is precisely that used to construct R from the Cauchy sequences in Q, but it presupposes that R has already been constructed. In a more general theory it can be shown that these two constructions of complete spaces are particular cases of the same general theorem.

Theorem 65.1. *Let (E, d) be a metric space. Then a metric space $(\hat{E}, \delta)$ can be constructed with the following properties:*

1. *There is a biuniform correspondence between E and a subset of $\hat{E}$*
2. *If we identify E and this subset of $\hat{E}$, the distance induced by δ on E in d, and E in dense in $\hat{E}$.*
3. *$(\hat{E}, \delta)$ is complete*

$(\hat{E}, \delta)$ is called the completion of (E, d).

Proof. Let Γ be the set of Cauchy sequences in (E, d). An element u of Γ is thus a sequence (x_n) of elements of E such that $\lim d(x_p, x_q) = 0$. Let $\mathscr{R}$ be the relation between elements $u = (x_n)$ and $v = (y_n)$ of Γ defined by

$$u \sim v \Leftrightarrow \lim d(x_n, y_n) = 0$$

$\mathscr{R}$ is clearly an equivalence relation. Let $\hat{E} = \Gamma/\mathscr{R}$. An element ξ of $\hat{E}$ is thus the set of all Cauchy sequences in (E, d) which are equivalent, modulo $\mathscr{R}$, to a given Cauchy sequence u.

Distance on $\hat{E}$

Let $u = (x_n)$, $v = (y_n)$ be two Cauchy sequences in (E, d) and let $\rho_n = d(x_n, y_n)$. We have

$$|d(x, y) - d(x', y')| \leqslant d(x, x') + d(y, y')$$

so that

$$|\rho_p - \rho_q| \leqslant d(x_p, x_q) + d(y_p, y_q)$$

Since (x_n) and (y_n) are Cauchy sequences, $\rho_p - \rho_q$ tends to zero, so that (ρ_n) is a Cauchy sequence in R and therefore converges. Let

$$\rho = \lim d(x_n, y_n) \geqslant 0$$

We shall show that ρ depends only on the equivalence classes ξ, η of (x_n) and (y_n). Let $(x'_n) \sim (x_n)$ and $(y'_n) \sim (y_n)$, then

$$|d(x_n, y_n) - d(x'_n, y'_n)| \leqslant d(x_n, x'_n) + d(y_n, y'_n)$$

and since

$$\lim d(x_n, x'_n) = \lim d(y_n, y'_n) = 0$$

it follows that

$$\lim d(x'_n, y'_n) = \lim d(x_n, y_n) = \rho$$

Put $\rho = \delta(\xi, \eta)$, then δ is a distance on $\hat{E}$. For $\rho \geqslant 0$; $\rho = 0$ implies that, for a sequence $(x_n) \in \xi$ and a sequence $(y_n) \in \eta$, we have

$$\lim d(x_n, y_n) = 0$$

so that $(x_n) \sim (y_n)$ which means $\xi = \eta$. Conversely, it is clear that if $\xi = \eta$ then

$$\delta(\xi, \eta) = 0$$

The symmetry property $\delta(\xi, \eta) = \delta(\eta, \xi)$ is evident.
Finally, from

$$d(x_n, y_n) \leqslant d(x_n, z_n) + d(z_n, y_n)$$

where (x_n), (y_n), (z_n) are Cauchy sequences in (E, d), there follows, with the obvious notation,

$$\delta(\xi, \eta) \leqslant \delta(\xi, \zeta) + \delta(\zeta, \eta)$$

for all $\xi,\ \eta,\ \zeta\ \in\ E$

(1) *Biuniform correspondence between* E *and a subset of* $\hat{E}$

To each $x \in E$ we assign in $\hat{E}$ the class ξ defined by the Cauchy sequence (x_n) such that $x_n = x$ for all n. Let A be the set of these equivalence classes, a subset of $\hat{E}$. If η is the image in A of $y \in E$ under the preceding mapping we have $\xi = \eta$ if and only if $(x) \sim (y)$, and so $\lim d(x, y) = 0$. Thus $d(x, y) = 0$ and $x = y$. This establishes a biuniform correspondence between E and $A \subset \hat{E}$.

(2) E *is dense in* $\hat{E}$

Let ξ, η be the images of x, y in $\hat{E}$. We have

$$\delta(\xi, \eta) = \lim d(x, y) = d(x, y)$$

This shows that the distance induced by δ on A is the distance d. We can therefore identify A and (E, d), which can be considered as a metric subspace of $(\hat{E}, \delta)$.

We now show that E is dense in $\hat{E}$. Let ξ, the equivalence class os a Cauchy sequence (x_n), be an element of $\hat{E}$. For each n let ξ_n be the class of the sequences $(x_{n,p})_{p \in N}$, where $x_{n,p} = x_n$ for all p. We have

$$\delta(\xi, \xi_n) = \lim_{p \to \infty} d(x_p, x_{n,p}) = \lim_{p \to \infty} d(x_p, x_n)$$

Now

$$\lim_{p,n\to\infty} d(x_p, x_n) = 0$$

so that $d(x_p, x_n) \leqslant \varepsilon$ if p and n are greater than or equal to $P(\varepsilon)$. It follows that

$$\lim_{p\to\infty} d(x_p, x_n) < \varepsilon$$

if $n \geqslant P(\varepsilon)$. Since ε is arbitrary this implies that

$$\lim_{n\to\infty} \delta(\xi, \xi_n) = 0$$

Note that we have proved that if (x_p) has a limit ξ in $(\hat{E}, \delta)$

$$\lim_{p\to\infty} \delta(\xi, x_p) < \varepsilon$$

$$\lim_{p,q\to\infty} \delta(x_p, x_q) = 0$$

and

$$\lim_{p\to\infty}(\lim_{q\to\infty d(x_p} , x_q)) = 0$$

(3) $(\hat{E}, \delta)$is complete.
Proof of the statement (3) is left as a homework.

$\square$