MA651 Topology. Lecture 12. Mapping in Metric Spaces.

This text is based on the following books:

- *"Introduction to Real Analysis" by A.N. Kolmogorov and S.V. Fomin*
- *"Linear Algebra and Analysis" by Marc Zamansky*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

66 Uniform continuity

The definition of continuity of a mapping f of a matric space (E, d) into a metric space (F, δ) is the same as that given in the case of two arbitrary topological spaces. However we can now define the concept of *uniform continuity.*

Definition 66.1. Let f be a mapping of a metric space (E, d) into a metric space (F, δ). f is said to be uniformly continuous if for each $\varepsilon > 0$ we can find $\alpha(\varepsilon) > 0$ such that if $d(x, x') < \alpha$ then $\delta(f(x), f(x')) < \varepsilon$.

Example 66.1. *If f is uniformly continuous it is continuous, but the converse is false. For example, the continuous mapping of R^+ into R^+ defined by $x \to 1/x$.*

There are cases where continuity implies uniform continuity. These depend on topological properties of the space (E, d). For example, let us prove the following result:

Theorem 66.1. *Let f be a mapping of a compact metric space (E, d) into a metric space (F, δ). if f is continuous it is uniformly continuous.*

Proof. Let $\varepsilon > 0$. To each $x \in E$ we assign an open ball $B(x, r_x)$ of center x and radius r_x such that if $x' \in B(x, r_x)$, $\delta(f(x'), f(x)) < \varepsilon/2$. This may be done since f is continuous.

Consider the open balls $B(x, r_x/2)$. They cover E and, since E is compact, include a finite cover $B(x_i, r_{x_i}/2)$.

Let

$$m = \inf(r_{x_i}/2)$$

and consider two points x, x' of E such that $d(x, x') < m$. The point x is contained in a certain ball $B(x_i, r_{x_i}/2)$ and we have

$$d(x', x_i) \leqslant d(x', x) + d(x, x') < m + r_{x_i}/2 \leqslant r_{x_i}$$

It follows that $x' \in B(x_i, r_{x_i})$. We now have

$$\delta(f(x'), f(x)) < \delta(f(x'), f(x_i)) + \delta(f(x), f(x_i)) < \varepsilon$$

since x' and x belong to $B(x_i, r_{x_i})$.

□

Example 66.2. *Let us consider a continuous function defined by distance. The distance $d(x, A)$ of a point x from a subset A of a metric space E is defined by*

$$d(x, A) = \inf_{y \in A} d(x, y)$$

We will prove that the function $x \to d(x, A)$ is uniformly continuous on E for every set A.

$d(x, A)$ being the lower bound of the $d(x, u)$ for $u \in A$, given $\varepsilon > 0$ there exists a $u_0 \in A$ such that

$$d(x, A) \leqslant d(x, u_0) < d(x, A) + \varepsilon$$

Let y be another point of E. We have

$$d(y, u_0) \leqslant d(x, y) + d(x, u_0) < d(y, x) + d(x, A) + \varepsilon$$

and since

$$d(x, A) = \inf_{u \in A} d(y, u) \leqslant d(y, u_0)$$

it follows that

$$d(y, A) \leqslant d(y, x) + d(x, A) + \varepsilon$$

Since ε is arbitrary there results

$$d(y, A) \leqslant d(y, x) + d(x, A)$$

and interchanging x and y

$$d(x, A) \leqslant d(x, y) + d(y, A)$$

Thus for each subset A and arbitrary points x, y of E we have

$$|d(x, A) - d(y, A)| < d(x, y)$$

which establishes the uniform continuity of the function $x \to d(x, A)$.

67 Extension by continuity

The following question arises in a natural way. If E and F are two spaces, A a dense subspace of E, and ϕ a continuous mapping of A into F, is there a mapping f of E into F which is continuous in E and whose restriction to A is ϕ?

This question can be posed more graphically as follows: if f is a continuous mapping of E into F and A is dense in E can f be reconstituted from its restriction ϕ to A?

The solution of this problem is called the *extension of ϕ from A to E by continuity.* This, in fact, can be done if we impose some quite general conditions satisfied by metric spaces (which are separated and normal).

We first prove the following statement:

Proposition 67.1. If f and g are two continuous mappings of a space E into a separated space F, and are equal at the points of a dense subset A of E, then they are equal everywhere in E.

Proof. For, if $x \in E$ is adherent to A, $f(x)$, the limit of $f(\xi)$ when ξ tends to x, is also the limit when ξ tends to x in A. Since $f(\xi) = g(\xi)$ for $\xi \in A$, and F is separated, the limits of f and g at each point $x \in E$ are equal.

□

Now let ϕ be a mapping of a set A which is dense in E, into a separated space F. In order to be able to extend ϕ to E we must suppose that when $\xi \in A$ tends to $x \in E$, $\phi(\xi)$ has a limit, which we shall denote by $f(x)$. (If E is a metric space we can say that for every sequence (ξ_n) of elements of A converging to $x \in E$, $\phi(\xi_n)$ must have a limit, and this limit must be the same for every such sequence (ξ_n).) We now prove the following theorem:

Theorem 67.1. *Let A be a dense subspace of a space E, F a normal space, and ϕ a mapping of A into F such that for every $x \in E$, $\phi(\xi)$ has a limit $f(x)$ in F when $\xi \in A$ tends to x. Then the function f is continuous in E.*

Proof. See the proof of Theorem 38.1

□

68 Contraction mapping

68.1 The fixed point theorem

Let A be a mapping of a metric space R into itself. Then x is called a *fixed point* of A if $Ax = x$, i.e. A carries x into itself. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \leqslant \alpha\rho(x, y)$$

for every pair of points $x, y \in R$. Then A is said to be a *contraction mapping.* Every contraction mapping is automatically continuous, since it follows from the "contraction condition" $(\rho(Ax, Ay) \leqslant \alpha\rho(x, y))$ that $Ax_n \to Ax$ whenever $x_n \to x$.

Theorem 68.1. ***(Fixed point theorem)****. Every contraction mapping A defined on a complete metric space R has a unique fixed point.*

Proof. Given an arbitrary point $x_0 \in R$, let

$$x_1 = Ax_0,\ x_2 = Ax_1 = A^2x_0,\ \dots,\ x_n = Ax_{n-1} = A^n x_0, \dots$$

where $A^2x = A(A(x))$, $A^3x = A(A^2x) = A(A(A(x)))$, etc.
Then the sequence $\{x_n\}$ is fundamental. In fact, assuming to be explicit that $n \leqslant n'$, we have

$$\begin{aligned}
\rho(x_n, x_{n'}) &= \rho(A^n x_0, A^{n'} x_0) \leqslant \alpha^n \rho(x_0, x_{n'-n}) \\
&\leqslant \alpha^n[\rho(x_0, x_1) + \rho(x_1, x_2) + \dots + \rho(x_{n'-n-1}, x_{n'-n})] \\
&\leqslant \alpha^n \rho(x_0, x_1)[1 + \alpha + \alpha^2 + \dots + \alpha^{n'-n-1}] \\
&< \alpha^n \rho(x_0, x_1)\frac{1}{1-\alpha}
\end{aligned}$$

But the expression on the right can be made arbitrary small for sufficiently large n, since $\alpha < 1$. Since R is complete, the sequence $\{x_n\}$, being fundamental, has a limit

$$x = \lim_{n\to\infty} x_n$$

Then, by continuity of A,

$$Ax = A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x$$

This proves the existence of a fixed point x. To prove the uniqueness of x we note that is

$$Ax = x, \quad Ay = y$$

then by definition of contraction mapping

$$\rho(x, y) \leqslant \alpha\rho(x, y)$$

But then $\rho(x, y) = 0$ since $\alpha < 1$, and hence $x = y$ □

Remark. The fixed point theorem can be used to prove existence and uniqueness theorems for solutions of equations of various types. Besides showing that an equation of the form $Ax = x$ has a unique solution, the fixed point theorem also gives a practical method for finding the solution, i.e. calculation of the "successive approximations" ($x_n = A^n x_0$). In fact, as shown in the proof, the approximations actually converge to the solution of the equation $Ax = x$. For this reason, the fixed point theorem if often called the method of successive approximations.

Example 68.1. *Let f be a function defined on the closed interval $[a, b]$ which maps $[a, b]$ into itself and satisfies a Lipschitz condition*

$$|f(x_1) - f(x_2)| \leqslant K|x_1 - x_2| \tag{1}$$

with constant $K < 1$. Then f is a contraction map, and hence, by Theorem (68.1), the sequence

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1),\ \dots \tag{2}$$

converges to the unique root of the equation $f(x) = x$. In particular, "contraction condition" (1) holds if f has a continuous derivative f' on $[a, b]$ such that

$$|f'(x)| \leqslant K < 1$$

Example 68.2. *Consider the mapping A of $n-$dimensional space into itself given by the system of linear equations*

$$y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i \ \ (i = 1, \dots, n) \tag{3}$$

If A is a contraction mapping, we can use the method of successive approximations to solve the equation $Ax = x$. The condition under which A is a contraction mapping depend on the choice of metric. We now examine three cases:

1. *The space R_0^n with metric*

$$\rho(x, y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|$$

In this case,

$$\begin{aligned}
\rho(y, \widetilde{y}) = \max_i |y_i - \widetilde{y}_i| &= \max_i |\sum_j a_{ij}(x_j - \widetilde{x}_j)| \\
&\leqslant \max_i \sum_j |a_{ij}||(x_j - \widetilde{x}_j)| \\
&\leqslant \max_i \sum_j |a_{ij}| \max_j |(x_j - \widetilde{x}_j)| = (\max_i \sum_j |a_{ij}|)\rho(x, \widetilde{x})
\end{aligned}$$

and the contraction condition is now

$$\sum_j |a_{ij}| \leqslant \alpha < 1 \quad (j = 1, \ldots, n) \tag{4}$$

2. The space R_1^n with metric

$$\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

Here

$$\begin{aligned}\rho(y, \widetilde{y}) &= \sum_i |x_i - \widetilde{y}_i| = \sum_i |\sum_j a_{ij}(x_j - \widetilde{x}_j)| \\ &\leqslant \sum_i \sum_j |a_{ij}||(x_j - \widetilde{x}_j)| \\ &\leqslant (\max_j \sum_i |a_{ij}|)\rho(x, \widetilde{x})\end{aligned}$$

and the contraction condition is now

$$\sum_j |a_{ij}| \leqslant \alpha < 1 \quad (j = 1, \ldots, n) \tag{5}$$

3. Ordinary Euclidean space R^n with metric

$$\rho(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Using the Cauchy-Schwarz inequality, we have

$$\rho^2(y, \widetilde{y}) = \sum_i (\sum_j a_{ij}(x_j - \widetilde{x}_j))^2 \leqslant (\sum_i \sum_j a_{ij}^2)\rho^2(x, \widetilde{x})$$

and the contraction condition becomes

$$\sum_i \sum_j a_{ij}^2 \leqslant \alpha < 1 \tag{6}$$

Thus, if at least one of conditions (4-6) holds, there exist a unique point $x = (x_1, x_2, \ldots, x_n)$ such that

$$x_i = \sum_{i=1}^{n} a_{ij} x_j + b_i \ \ (i = 1, \ldots, n)$$

The sequence of successive approximations to this solution of the equation $x = Ax$ are of the form

$$\begin{aligned} x^0 &= (x_1^0, x_2^0, \ldots, x_n^0) \\ x^1 &= (x_1^1, x_2^1, \ldots, x_n^1) \end{aligned}$$

$$x^k = (x_1^k, x_2^k, \ldots, x_n^k)$$

where

$$x_i^k = \sum_{i=1}^{n} a_{ij} x_j^{k-1} + b_i$$

and we can choose any point x^0 as the "zeroth approximation".
Each of the conditions (4-6) is sufficient *for applicability of the method of successive approximations, but none of them is* necessary. *In fact, examples can be constructed in which each of the conditions (4-6) is satisfied, but not the other two.*

68.2 Contraction mapping and differential equations

The most interesting applications of Theorem (68.1) arise when the space R is a function space. We can use this theorem to prove a number of existence and uniqueness theorems for differential and integral equations.

Theorem 68.2. ***(Picard)***. *Given a function $f(x, y)$ defined and continuous on a plane domain G containing the point (x_0, y_0) suppose f satisfies a Lipschitz condition of the form*

$$|f(x, y) - f(x, \widetilde{y})| \leqslant M|y - \widetilde{y}|$$

in the variable y. Then there is an interval $|x - x_0| \leqslant \delta$ in which the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{7}$$

has a unique solution

$$y = \varphi(x)$$

satisfying the initial condition

$$\varphi(x_0) = y_0 \tag{8}$$

Remark. By an $n-$dimensional *domain* we mean an open connected set in Euclidean $n-$space.

Proof. Together the differential equation (7) and the initial condition (8) are equivalent to the integral equation

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt \tag{9}$$

By the continuity of f, we have

$$|f(x, y)| \leqslant K \tag{10}$$

in some domain $G' \subset G$ (in fact f is bounded on $\overline{G'} \subset G$) containing the point (x_0, y_0). Choose $\delta > 0$ such that

1. $(x, y) \in G'$ if $|x - x_0| \leqslant \delta$, $|y - y_0| \leqslant K\delta$
2. $M\delta < 1$

and let C^* be the space of continuous functions φ defined on the interval $|x - x_0| \leqslant \delta$ and such that $|\varphi(x) - y_0| \leqslant K\delta$, equipped with the metric

$$\rho(\varphi, \widetilde{\varphi}) = \max_x |\varphi(x) - \widetilde{\varphi}(x)|$$

The space C^* is complete, since it is closed subspace of the space of all continuous functions on $[x_0 - \delta, x_0 + \delta]$. Consider the mapping $\psi = A\varphi$ defined by the integral equation

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt \quad (|x - x_0| \leqslant \delta)$$

Clearly A is a contraction mapping carrying C^* into itself. In fact, if $\varphi \in C^*, |x - x_0| \leqslant \delta$ then

$$\begin{aligned} |\psi(x) - y_0| = | \int_{x_0}^{x} f(t, \varphi(t))dt| &\leqslant \\ \leqslant \int_{x_0}^{x} |f(t, \varphi(t))|dt &\leqslant \\ \leqslant K|x - x_0| \leqslant K\delta \end{aligned}$$

by (10), and hence $\psi = A\varphi$ also belongs to C^*. Moreover,

$$\begin{aligned} |\psi(x) - \widetilde{\psi}(x)| &\leqslant \int_{x_0}^{x} |f(t, \varphi(t)) - f(t, \widetilde{\varphi}(t))|dt \\ &\leqslant M\delta|\varphi(t) - \widetilde{\varphi}(t)| \end{aligned}$$

and hence

$$\rho(\psi, \widetilde{\psi}) \leqslant M\delta\rho(\psi, \widetilde{\psi})$$

after maximizing with respect to x. But $M\delta < 1$, so that A is a contraction mapping. It follows from Theorem (68.1) that the equation $\varphi = A\varphi$, i.e. the integral equation (9), has a unique solution in the space C^*. $\square$