MA651 Topology. Lecture 3. Topological spaces.

This text is based on the following books:

- *"Linear Algebra and Analysis" by Marc Zamansky*
- *"Topology" by James Dugundgji*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

A topological space has a structure in which the concepts of limit and continuous function can be specified.Though R^1 and R^n are obviously different, this is not due to one having more points than the other, since we know that card $R^1 =$ card R^n. Geometrically, it is evident that the points are arranged differently, so that different subsets are "close together". To detect inherent differences of this sort, we study sets in which a notion of "nearness" is specified, that is, in which topology is specified.
All the fundamental concepts of topology derive from properties of real numbers and real-valued functions of a real variable. However, the real numbers have a very rich structure; their properties stem from many fundamental notions: total order relations, group, ring and field properties, the concept of an absolute value, the existence of the rational numbers forming a dense set, the fact that every Cauchy sequence converges, etc.
If we consider, for example, continuous functions, we see that some properties are true when the variable lies in an interval (open or not), and that others are true only if the interval is closed. We have therefore tried to prove our theorems under the most general hypotheses, and are led to formulate the basic concepts required for their proof.
The concept of a continuous function enables us to give a sense to the expression "$f(x)$ tends to y_0 when x tends to x_0". This concept of a limit proves inadequate. Further, the concept of a countable convergent sequence which is the appropriate concept in the study of metric spaces proves inadequate for non-metric spaces. Finally, we find it desirable to have a unified way of describing such diverse phenomena as the following: $f(x)$ tends to y_0 when x tends to x_0, $f(x)$ tends to y_0 when x tends to x_0 on the right (or on the left), x_n tends to y_0 when n tends to infinity,

situations which are not always related to a concept of continuity.
We arrive at axiomatic definitions of open sets, neighborhoods, filter bases, etc. This method, although perhaps unfamiliar, has the advantage of enabling us to derive certain properties from the minimum hypotheses. These concepts have their origins, directly or indirectly, in familiar notions taken from the real line. The usual way of developing of the subject consists in defining open sets, then neighborhoods of a point, or firstly neighborhoods and then open sets; the general concept of a limit is introduced afterwards, if needed.
Now the principal concepts taken from properties of the real line depend on the fact that we use open intervals (we can even consider only intervals whose points are rational numbers), and if we consider the set of open intervals it is clear that, if X and Y are two such intervals, their intersection $X \bigcap Y$ contains another open interval (the empty set being also considered as an open interval).
This observation is even more illuminating in the case of the plane. When we study, for example, the concept of continuity or limit in the plane we need only make use of open discs. Now the intersection of two open discs is either empty, or is a set which, although not itself an open disc, contains one. In our development of the subject this property, elevated to the status of an axiom, plays a fundamental role.

19 "Local" Definition of Topology

In this section we define topology starting from elements of a set E, while in the next section we will define topology as a collection of subsets of E, in fact, both definitions are equivalent.

19.1 Fundamental Families

The fundamental idea consist in the fact that in questions of topology or of limits we have to do with a family Ω of subsets X of set E such that the intersection $X \cap X'$ of two elements of Ω contains another element X'' of Ω. However if we want to use this observation as an axiom we must note that the empty set $\emptyset \in \Omega$, since for every subset A of E we have $\emptyset \subset A$, the condition that $X \cap X'$ contains X'' will always be satisfied by taking $X'' = \emptyset$. We shall have to make our condition more precise.
On the other hand the empty set will play an important part. In fact, in broad terms, the property $\emptyset \notin \Omega$ is related to the concept of limit, and the property $\emptyset \in \Omega$ is related to that of topology.
We agree, once and for all, that we shall not consider empty families.

Definition 19.1. A fundamental family on a set E is a non-empty family Ω of subsets of E such that if X and X' belong to Ω, $X \cap X'$ contains an element X'' of Ω, and $X'' \neq \emptyset$ if $X \cap X' \neq \emptyset$.

This definition gives rise to the following observations:

1. If Ω is a fundamental family on a set E, if it contains non-empty sets, and $\emptyset \in \Omega$, it need not contain non-empty sets X, X' such that $X \cap X' = \emptyset$. For example, we may take Ω to consist of $\emptyset$ and E.

2. If a non-empty family Ω is a fundamental family, and if $\emptyset \notin \Omega$, then no element of Ω is empty, and two (and so any finite number of) arbitrary elements of Ω have non-empty intersection.

3. If Ω is a given family not containing $\emptyset$, to prove that Ω is a fundamental family it suffices to show that for two *arbitrary* elements X, X', $X \cap X'$ contains an $X'' \in \Omega$.

4. If Ω is a family containing two non-empty elements X, X' such that $X \cap X' = \emptyset$, in order that Ω shall be a fundamental family we must have $\emptyset \in \Omega$.

5. The more restrictive condition $X \cap X' \in \Omega$ also defines a fundamental family.

6. A family consisting of a single subset of a set is a fundamental family, but is without interest.

7. If Ω is a fundamental family which does not contain $\emptyset$, we again have a fundamental family on adjoining $\emptyset$ to Ω.

19.2 Properties of Fundamental Families

1. If Ω is a fundamental family, every finite intersection of its members contains an element of Ω, and this element is not empty if the intersection is not empty.

2. Let Ω be a fundamental family on a set E, and f a mapping of E into a set F. Let $\mathscr{Y}$ be the family of subsets of F consisting of the $f(X)$ where $X \in \Omega$.

   (a) In general $\mathscr{Y}$ is not a fundamental family. We give an example of this.
   Let A, A' be *disjoint* intervals of R, and B, B' two *distinct non-disjoint* intervals of R. Using restrictions of linear maps

   $$(f(x) = \alpha x + \beta \text{ on } A \textit{ or } A')$$

   we can define mapping f such that $B = f(A)$, $B' = f(A')$. Let Ω be the fundamental family consisting of $\emptyset$, A, A'. Then $\mathscr{Y}$ consists of $\emptyset$, B, B'.
   Ω is a fundamental family since $A \cap \emptyset = A' \cap \emptyset = A \cap A' = \emptyset \in \Omega$. However $\mathscr{Y}$ is not a fundamental family since $B \cap B'$ is not one of $\emptyset$, B and B'.

   (b) Now suppose that $\emptyset \notin \Omega$. If $X \neq \emptyset$ then $f(X) \neq \emptyset$, $\emptyset \notin \mathscr{Y}$. If $Y = f(X)$ and $Y' = f(X')$ are two elements of $\mathscr{Y}$, since $X \cap X' \neq \emptyset$, there exists a non-empty element X'' of Ω such that $X'' \subset X \cap X'$ and

   $$\emptyset \neq Y'' = f(X'') \subset f(X \cap X') \subset f(X) \cap f(X') = Y \cap Y'.$$

Whence:

Proposition 19.1. If Ω is a fundamental family on E, not containing $\emptyset$, the images, be a mapping f of E into a set F, of the elements of Ω form a fundamental family on F not containing $\emptyset$.

Briefly: the image of a fundamental family not containing $\emptyset$ is a fundamental family.

3. Let A be a non-empty subset of E, Ω a fundamental family on E, $\mathscr{Y}$ the set of $Y = X \cap A$ where $X \in \Omega$ (Y is called the *trace* of X on A and $\mathscr{Y}$ the *trace* of Ω on A. We interested in conditions when $\mathscr{Y}$ is a fundamental family.
In general, the trace of a fundamental family is not a fundamental family.
Suppose that for every non-empty X in Ω, $X \cap A \neq \emptyset$, i.e. *every non-empty X meets A.*

Proposition 19.2. If Ω is a fundamental family on E and if every non-empty $X \in \Omega$ meets a non-empty subset A of E the trace of Ω on A is a fundamental family.

and as a consequence we have

Proposition 19.3. Let f be a mapping of E into F, $\mathscr{Y}$ a fundamental family on E. If every non-empty Y of $\mathscr{Y}$ meets $f(E)$, the inverse image of a fundamental family on F is a fundamental family on E.

19.3 Comparison of Fundamental Families

Definition 19.2. Let Ω, Ω' be two fundamental families on a set E, both either containing, or not containing the empty set $\emptyset$. The Ω' is said to be coarser than Ω (or Ω is finer than Ω') if every non-empty X' of Ω' contains a non-empty X of Ω

If Ω' is coarse than Ω, and Ω coarse than Ω' then Ω and Ω' are said *equivalent.*

19.4 Definition of a topological space

Definition 19.3. A set E is called a topological space if there is associated with every $x \in E$ a non-empty family $\mathscr{B}(x)$ of subsets of E satisfying the following conditions for each $x \in E$:

(1) $\mathscr{B}(x)$ is a fundamental family all whose elements contain x.

(2) If y belongs to $X \in \mathscr{B}(x)$ then X contains a $Y \in \mathscr{B}(y)$.

The family $\mathscr{B}(x)$ is called a *basis (base) for the open neighborhoods of x*, and the members of $\mathscr{B}(x)$ are called *open neighborhoods* of x.
When this is done we say that we have defined a *topology* or a *topological structure* on E.

20 "Global" Definition of Topology

Definition 20.1. Let X be a set. A topology (or topological structure) in X is a family $\mathscr{T}$ of subsets of X that satisfies:

1. Each union of members of $\mathscr{T}$ is also a member of $\mathscr{T}$.
2. Each *finite* intersection of members of $\mathscr{T}$ is also a member of $\mathscr{T}$.
3. $\emptyset$ and X are members of $\mathscr{T}$.

Definition 20.2. A couple $(X, \mathscr{T})$ consisting of a set X and a topology $\mathscr{T}$ in X is called a *topological space.*

Instead of "topological space $(X, \mathscr{T})$" we also say "$\mathscr{T}$ is the topology of the space X", or "X carries topology $\mathscr{T}$". When it is not necessary to specify $\mathscr{T}$ explicitly, we simply say, "X is a space" (to distinguish from "X is a set").
Elements of topological spaces are called *points.* The members of $\mathscr{T}$ are called "open sets" of the topological space $(X, \mathscr{T})$ (or of the topology $\mathscr{T}$). There is no preconceived idea of what "open" means, other than the sets called open in any discussion satisfy the axioms $1-3$ of Definition (20.1). Observe that since the union (respectively intersection) of an empty family of sets in X is $\emptyset$ (respectively X), Axiom 3 of of Definition (20.1) is actually redundant.

Definition 20.3. Let $(X, \mathscr{T})$ be a space. By a neighborhood (written $U(x)$) of an $x \in X$ is meant any open set (that is, member of $\mathscr{T}$) containing x.

We will consider the points of $U(x)$ to be "U-close" to x, so that $\mathscr{T}$ organizes X into chunks of "nearby" points. As the following examples show, a set X may have many topologies; with each it is a distinct topological space.

Example 20.1. *Let X be any set; $\mathscr{I} = \{\emptyset, X\}$. This topology, in which no set other than $\emptyset$ and X is open, is called the* indiscrete topology *or* trivial topology *$\mathscr{I}$. In this case there is no "small neighborhood".*

Example 20.2. *In the set X, let $\mathscr{T} = \mathscr{P}(X)$. This is called the* discrete topology *$\mathscr{D}$, where every set is an open set.*

Example 20.3. *The topological space consisting of the two points $\{0, 1\}$ with the discrete topology is denoted by 2. The same set with the topology $\mathscr{J} = \{\emptyset, 0, X\}$ is called the Sierpinski space $\mathscr{J}$. In contrast to the space 2, 1 has no "small" neighborhood in $\mathscr{J}$.*

Example 20.4. *Let R be the set of real numbers. In the usual introduction to analysis, a subset $G \subset R$ is called "open" if for each $x \in G$ there is an $r > 0$ such that the symmetric open interval $B(x; r) = \{y \mid |y - x| < r\} \subset G$. We verify that the family $\mathscr{T}$ of sets declared "open" by this criterion actually is topology in the set R:*

1. *Axiom 1. If each member of* $\{G_\alpha \mid \alpha \in \mathscr{A}\}$ *is "open", so also is* $\bigcup_\alpha G_\alpha$*, since* $x \in \bigcup_\alpha G_\alpha \Rightarrow (\exists\, \alpha : x \in G_\alpha) \Rightarrow (\exists\, r > 0 : B(x;r) \subset G_\alpha \subset \bigcup_\alpha G_\alpha)$.

2. *Axiom 2. If* $G_1, \cdots, G_n$ *are "open", so also is* $\bigcap_1^n G_i$ *because* $x \in \bigcap_1^n G_i \Rightarrow (\forall\, i : x \in G_i) \Rightarrow (\forall\, i\, \exists\, r_i > 0 : B(x;r_i) \subset G_i) \Rightarrow [B(x; min(r_1, \cdots, r_n)) \subset \bigcap_1^n G_i]$.

3. *Axiom 3 is trivial.*

This topology $\mathscr{T}$*, is called the* Euclidean topology *of R; the topological space* $(R, \mathscr{T})$ *is called the Euclidean 1-space. Note that* $\mathscr{T}$ *is* not *the indiscrete topology in the set R, that is,* $\mathscr{T}$ *does not consist only of* $\emptyset$ *and R: indeed, each* $B(y;r)$ *belongs to* $\mathscr{T}$*, since, given any* $x \in B(y;r)$*, we have* $d = |y - x| < r$*, and therefore* $x \in B(x; r-d) \subset B(y;r)$*. It is simple to see that* $\mathscr{T}$ *can also be described more directly as the family of all unions of open intervals.*

Example 20.5. *Let* R^n *be the set of all ordered n-uples of real numbers. Using vector notation* $(x = (x_1, \cdots x_n))$ *call " ball of center x and radius r" the set* $B(x;r) = \{y \mid |y - x| < r\}$*. The* Euclidian topology *in* R^n *is determined by calling* $G \subset R^n$ *"open" if for each* $x \in G$ *there is some* $r > 0$ *such that* $B(x;r) \subset G$*. The verification that this criterion describes a topology* $\mathscr{T}$*, and that in fact each* $B(x;r)$ *belongs to* $\mathscr{T}$*, is contained in Example (20.4), since only reinterpretation of* $B(x;r)$ *is involved. With this topology,* R^n *is called* Euclidean n-space*. As before, the open sets in* R^n *can be described equally as the arbitrary unions of balls.*

By regarding each topology as a subset of $\mathscr{P}(X)$, the topologies in X are partially ordered by the inclusion; clearly $\mathscr{I} \subset \mathscr{T} \subset \mathscr{D}$ for each topology $\mathscr{T}$. We call $\mathscr{T}_1$ larger (or "with more open sets") than $\mathscr{T}_0$; and $\mathscr{T}_0$ smaller than $\mathscr{T}_1$ whenever $\mathscr{T}_0 \subset \mathscr{T}_1$, and more formally:

Definition 20.4. Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies in X. If $\mathscr{T} \subset \mathscr{T}'$, we say that $\mathscr{T}'$ is *larger (finer)* than $\mathscr{T}$. And $\mathscr{T}$ is smaller (coarser) than $\mathscr{T}'$. If $\mathscr{T}'$ properly contains $\mathscr{T}$, $\mathscr{T} \subsetneqq \mathscr{T}'$, we say that $\mathscr{T}$ is *strictly coaster* than $\mathscr{T}'$. We say that $\mathscr{T}$ is *compatible* with $\mathscr{T}'$ if either $\mathscr{T} \subset \mathscr{T}'$ or $\mathscr{T}' \subset \mathscr{T}$.

The following proposition is trivial to verify:

Proposition 20.1. Let $\{\mathscr{T}_\alpha \mid \alpha \in \mathscr{A}\}$ be any family of topologies in X. Then $\bigcap_\alpha \mathscr{T}_\alpha = \{U \mid \forall\alpha \in \mathscr{A} : U \in \mathscr{T}_\alpha\}$ is also a topology in X; however, $\bigcup_\alpha \mathscr{T}_\alpha$ need not be a topology.

21 Basis for a Given Topology

The task of specifying topology is simplified by giving only enough open sets to "generate" all the open sets.

Definition 21.1. Let $(X, \mathscr{T})$ be a topological space. A family $\mathscr{B} \subset \mathscr{T}$ is called a basis for $\mathscr{T}$ if each open set (that is, member of $\mathscr{T}$) is the *union* of members of $\mathscr{B}$.

$\mathscr{B}$ is also called a "basis for the space X", and its members the "basic open set of the topology $\mathscr{T}$". Not only is each member of $\mathscr{T}$ the union of members of $\mathscr{B}$, but also, because of $\mathscr{B} \subset \mathscr{T}$ and Axiom (1) of Definition (20.1), each union of members of $\mathscr{B}$ belongs to $\mathscr{T}$; thus a basis for $\mathscr{T}$ completely determines $\mathscr{T}$.

Example 21.1. *$\mathscr{T}$ is a basis for $\mathscr{T}$.*

Example 21.2. *Let $\mathscr{D}$ be the discrete topology on X. Then the collection of all one-point subsets of X, $\mathscr{B} = \{\{x\} \mid x \in X\}$ is a basis for $\mathscr{D}$.*

In view of examples (21.1) and (21.2), a given $\mathscr{T}$ may have many bases. The families $\mathscr{B} \subset \mathscr{T}$ that can serve as a basis are characterized by

Theorem 21.1. *Let $\mathscr{B} \subset \mathscr{T}$. The following two properties of $\mathscr{B}$ are equivalent:*

1. *$\mathscr{B}$ is a basis for $\mathscr{T}$.*
2. *For each $G \in \mathscr{T}$ and each $x \in G$ there is a $U \in \mathscr{B}$ with $x \in U \subset G$.*

Proof.

- (1) $\Rightarrow$ (2). Let $x \in G$; since $G \in \mathscr{T}$ and $\mathscr{B}$ is a basis, $G = \bigcup\limits_{\alpha} U_\alpha$, where each $U_\alpha \in \mathscr{B}$. Thus there is at least one $U_\alpha \in \mathscr{B}$ with $x \in U_\alpha \subset G$.
- (2) $\Rightarrow$ (1). Let $G \in \mathscr{T}$; for each $x \in G$, find $U_x \in \mathscr{B}$ with $x \in U_x \subset G$; then $G = \bigcup\{U_x \mid x \in G\}$.

$\square$

Example 21.3. *In each R^n, $n \geqslant 1$, $\mathscr{B} = \{B(x; r) \mid x \in R^n, r > 0\}$ is a basis for the Euclidean topology, as the descriptions in examples (20.4),(20.5) show.*

Example 21.4. *R^n has a* countable basis*: the family $\mathscr{B} = \{B(\xi, r) \mid \xi$ has all coordinates rational, and $r > 0$ is rational $\}$. Let G be any open set, and $x \in G$. By examples (20.4),(20.5), there is a $B(x, r) \subset G$, and we can clearly assume that r is rational. Obviously there is a point ξ, with all coordinates rational, within a distance $\frac{r}{s}$ of x; then $x \in B(\xi, \frac{x}{2}) \subset B(x, r) \subset G$ as required by 2.2. Immediate consequences are:*

(a) Each set open in R^n is the union of at most countably many balls.

(b) The cardinal number of the topology of R^n is $2^{\aleph_0}$.

By specifying a basis for $\mathscr{T}$, all the open sets are generated as unions. However, there is a more convenient way to describe the open sets:

Theorem 21.2. *Let $\mathscr{B} \subset \mathscr{T}$ be a basis for $\mathscr{T}$. Then A is open (that is, is in $\mathscr{T}$) if and only if for each $x \in A$ there is a $U \in \mathscr{B}$ with $x \in U \subset A$.*

Proof. If A is open the condition follows from Theorem (21.1). Conversely, if the condition holds, then (as in Theorem (21.1)) we find $A = \bigcup\{U_a \mid a \in A\}$, where each $U_a \in \mathscr{B} \subset \mathscr{T}$; from Definition (20.1) follows that A is open. □

22 Topologizing of Sets

In this section, two general methods for introducing topologies in sets will be given.
The first, and most popular, starts from *any* given family $\Sigma \subset \mathscr{P}(X)$ and leads to a unique topology containing Σ.

Theorem 22.1. *Given* any *family $\Sigma = \{A_\alpha \mid \alpha \in \mathscr{A}\}$ of subsets of X, there always exists a unique, smallest topology $\mathscr{T}(\Sigma) \subset \Sigma$. The family $\mathscr{T}(\Sigma)$ can be described as follows: It consists of $\emptyset$, X, all finite intersections of the A_α, and all arbitrary unions of these finite intersections. Σ is called a subbasis for $\mathscr{T}(\Sigma)$, and $\mathscr{T}(\Sigma)$ is said to be generated by Σ.*

Proof. Let $\mathscr{T}(\Sigma)$ be the intersection of all topologies containing Σ; such topologies exist, since $\mathscr{P}(X)$ is one such. By Proposition (20.1) $\mathscr{T}(\Sigma)$ is a topology; it evidently satisfies the requirements "unique" and "smallest". To verify that the members of $\mathscr{T}(\Sigma)$ are as described, note that since $\Sigma \subset \mathscr{T}(\Sigma)$, it follows from Definition (20.1) that $\mathscr{T}(\Sigma)$ must contain all the sets listed. Conversely, because $\bigcup_\alpha$ distributes over $\cap$, the sets listed actually do form a topology containing Σ, and which therefore contains $\mathscr{T}(\Sigma)$. □

In Theorem (22.1), we started from Σ and obtained topology $\mathscr{T}(\Sigma) \supset \Sigma$. If, conversely, we are given a topology $\mathscr{T}$, a family $\Sigma \subset \mathscr{T}$ is called a subbasis for $\mathscr{T}$ whenever $\mathscr{T} = \mathscr{T}(\Sigma)$.

Example 22.1. *For any topology $\mathscr{T}$, $\mathscr{T}$ is a subbasis for $\mathscr{T}$.*

Example 22.2. *The* finite *intersections of members of Σ are a* basis *for $\mathscr{T}(\Sigma)$.*

Example 22.3. *In the set R of all real numbers, let Σ be all sets of from $\{x \mid x > a\}$ and $\{x \mid x < b\}$. Then $\mathscr{T}(\Sigma)$ is precisely the Euclidean topology: Each finite open interval, being an intersection of two subbasic open sets, belongs to $\mathscr{T}(\Sigma)$, and it is evident from the description of $\mathscr{T}(\Sigma)$ in Theorem (22.1)that the family $\mathscr{B}$ of all these finite intervals forms a basis for $\mathscr{T}(\Sigma)$. But by Example (21.3), $\mathscr{B}$ is a basis for the Euclidean topology.*

The construction of topology from a subbasis loses some control over the open sets; they build up from the finite intersections of the A_α rather from the A_α themselves. In the second general method for topologizing a set, which we will now describe, the open sets are constructed only by union from the given family; that is, by specifying a family to be used as a *basis* for constructing the topology. Since intersections are involved in topologies but not intersections are involved in forming open sets from a basis, it is to be expected that not every family can serve as the basis for *some* topology.

Theorem 22.2. *Let $\mathscr{B} = \{U_\mu \mid \mu \in \mathscr{M}\}$ be any family of subsets of X that satisfies the following condition:*

For each $(\mu, \lambda) \in \mathscr{M} \times \mathscr{M}$ and each $x \in U_\mu \cap U_\lambda$, there exists some U_α with $x \in U_\alpha \subset U_\mu \cap U_\lambda$.

Then the family $\mathscr{T}(\mathscr{B})$ consisting of $\emptyset$, X, and all unions of members of $\mathscr{B}$, is a topology for X; that is, $\mathscr{B}$ is a basis for some *topology. $\mathscr{T}(\mathscr{B})$ is unique and the smallest topology containing $\mathscr{B}$.*

Proof. Using Theorem (22.1), we obtain a topology $\mathscr{T}(\mathscr{B})$ having $\mathscr{B}$ as *subbasis*. To see that $\mathscr{T}(\mathscr{B})$ actually has $\mathscr{B}$ as a *basis*, we need show only that each finite intersection of members of $\mathscr{B}$ is in fact a union of members of $\mathscr{B}$. And, as in Theorem (21.1), (2) $\Rightarrow$ (1), it suffices to show that for each $x \in U_1 \cap \cdots \cap U_n$, there is a $U \in \mathscr{B}$ with $x \in U \subset U_1 \cap \cdots \cap U_n$. We proceed by induction, the assertion being true (by the hypothesis) for $n = 2$. If it is true for $(n-1)$, then writing $x \in U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$, the inductive hypothesis gives $x \in U \cup U_n$ for some

$$U \subset U_1 \cap \cdots \cap U_{n-1}$$

in $\mathscr{B}$ so, by the case $n = 2$, we find

$$x \in U' \subset U \cup U_n \subset U_1 \cap \cdots \cap U_n$$

for some $U' \in \mathscr{B}$, completing the induction and the proof of the theorem. □

The specification of a topology by giving a basis is generally accomplished by specifying for each $x \in X$ a family of neighborhoods $\{U_\alpha(x) \mid \alpha \in \mathscr{A}(x)\}$ (called a "fundamental family all of whose elements contain x", a "basis at x" or a "complete system of neighborhoods at x") and verifying that the family $\mathscr{B} = \{U_\alpha(x) \mid \alpha \in \mathscr{A}(x), x \in X\}$ satisfies the requirement of Theorem (22.2).

Though each basis in X gives a unique topology, it is obvious that distinct bases may give the same topology: For example, $\mathscr{T}(\mathscr{B})$ has $\mathscr{B}$ and $\mathscr{T}(\mathscr{B})$ as bases. We will now determine when this will occur.

Definition 22.1. Two bases $\mathscr{B}$, $\mathscr{B}'$ in X are *equivalent* if $\mathscr{T}(\mathscr{B}) = \mathscr{T}(\mathscr{B}')$.

Theorem 22.3. *A necessary and sufficient condition that two bases $\mathscr{B}$, $\mathscr{B}'$ in X be equivalent is that* both *the following conditions hold:*

1. *For each $U \in \mathscr{B}$ and each $x \in U$, there is a $U' \in \mathscr{B}'$ with $x \in U' \subset U$.*

2. *For each $U' \in \mathscr{B}'$ and each $x \in U'$, there is a $U \in \mathscr{B}$ with $x \in U \subset U'$.*

If only condition (1) *holds, then $\mathscr{T}(\mathscr{B})$ is a proper subset of $\mathscr{T}(\mathscr{B}')$.*

Proof. Assume $\mathscr{T}(\mathscr{B}) = \mathscr{T}(\mathscr{B}')$. Since each $U \in \mathscr{B} \subset \mathscr{T}(\mathscr{B})$ and $\mathscr{T}(\mathscr{B})$ has $\mathscr{B}'$ as basis, (1) follows from Theorem (21.1); similarly, (2) is true.

For the converse, assume (1) is true. Since each $V \in \mathscr{T}(\mathscr{B})$ is a union of sets belonging to $\mathscr{B}$, it follows from Theorem (21.2) that $V \in \mathscr{T}(\mathscr{B}')$, showing that $\mathscr{T}(\mathscr{B}) \subset \mathscr{T}(\mathscr{B}')$. If, in addition, (2) is true, we find $\mathscr{T}(\mathscr{B}') \subset \mathscr{T}(\mathscr{B})$, completing the proof.

□

23 Open and Closed Sets

Throughout this section, we consider a fixed topological space $(X, \mathscr{T})$, and give some definitions, all of which have familiar meaning when specialized to R.

Definition 23.1. $A \subset X$ is called *closed* if $\mathbf{C}_X A$ is an open set.

Proposition 23.1.

(a) The intersection of any family of closed sets is a closed set.

(b) The union of *finitely* many closed sets is a closed set.

Proof. These follow by De Morgan's rules; we prove only (a). To prove $\bigcap\limits_\alpha A_\alpha$ closed, we are to show that $\mathbf{C}\bigcap\limits_\alpha A_\alpha = \bigcup\limits_\alpha \mathbf{C}A_\alpha$ is open; however, since each $\mathbf{C}A_\alpha$ is open, so also is (by Definition (20.1) the union. □

The definition of closed set is not intrinsic, since we decide the question one considers the complement rather than the set. We will obtain some intrinsic formulations by using other concepts.

Definition 23.2. Let $A \subset X$. A point $x \in X$ is adherent point to A if each neighborhood of x contains at least one point of A (which may be x itself). The set $\overline{A} = \{x \in X \mid \forall U(x) : U(x) \cap A \neq \emptyset\}$ of all points in X adherent to A is called the closure of A.

Proposition 23.2.

(a) $A \subset \overline{A}$ for every set A.

(b) A is closed if and only if $A = \overline{A}$.

Proof.

(a) is immediate from Definition (23.2).

(b) (A closed) $\Rightarrow$ ($A = \overline{A}$): For, A closed $\Rightarrow$ $\mathbf{C}A$ is open, so each $x \notin A$ has a neighborhood (namely, $\mathbf{C}A$) not meeting A and therefore does not belong to $\overline{A}$. Thus $\overline{A} \subset A$ and with (a), $A = \overline{A}$.
($A = \overline{A}$) $\Rightarrow$ (A closed): For ($A = \overline{A}$) $\Leftrightarrow$ each $x \notin A$ has a neighborhood $U(x)$ not meeting A; by Theorem (21.2), this means that $\mathbf{C}A$ is open.

□

An alternative characterization of $\overline{A}$ and the algebraic properties of the closure operation are contained in

Proposition 23.3. $\overline{A}$ is the smallest closed set containing A; that is,

$$\overline{A} = \bigcap \{F \mid (F \text{ closed}) \wedge (F \supset A)\}.$$

Furthermore,

1. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$.
2. $\overline{\overline{A}} = \overline{A}$; that is, $\overline{A}$ is closed.
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. $\overline{\emptyset} = \emptyset$

Proof of this proposition is left as a homework.

Definition 23.3. Let $A \subset X$. A point $x \in X$ is called a cluster point of A if each neighborhood of x contains at least one point of A *distinct* from x. The set $A' = \{x \in X \mid \forall U(x) : U(x) \cup (A - x) \neq \emptyset\}$ of all cluster points of A is called the *derived* set of A.

Proposition 23.4. $\overline{A} = A \cup A'$. In particular, A is closed if and only if $A' \subset A$; that is, A contains all its cluster points.

Proof of this proposition is left as a homework.

Definition 23.4. Let $A \subset X$. the interior Int(A) of A is the largest open set contained in A; that is, $\text{Int}(A) = \bigcup\{U \mid (U \text{ open}) \wedge (U \subset A)\}$.

Proposition 23.5. $\text{Int}(A) = \mathbf{C}(\overline{\mathbf{C}A})$ for any set A. In particular, A is open if and only if $A =$Int(A).

Proof. Since $E \subset A \Leftrightarrow \mathbf{C}A \subset \mathbf{C}E$, we note that the *open* sets $E \subset A$ are precisely the complements of closed sets $F \supset \mathbf{C}A$. Thus

$$\begin{aligned}\text{Int}(A) &= \bigcup\{\mathbf{C}F \mid (F \text{ is closed}) \wedge (F \supset \mathbf{C}A)\}\\ &= \mathbf{C}\bigcup\{F \mid (F \text{ is closed} \wedge (F \supset \mathbf{C}A)\}\\ &= \mathbf{C}(\overline{\mathbf{C}A})\end{aligned}$$

by (23.3). The second part is now trivial. □

Definition 23.5. Let $A \subset X$. The boundary Fr(A) of A is $\overline{A} \cap \overline{\mathbf{C}A}$.

The boundary of A is a closed set, and both A, $\mathbf{C}A$ have the same boundary. The boundary can be described as the part of the closure not in the interior:

Proposition 23.6. Let $A \subset X$. Then:

1. $\text{Fr}(A) = \overline{A} - \text{Int}(A)$;
2. $\text{Fr}(A) \cap \text{Int}(A) = \emptyset$;
3. $\overline{A} = \text{Int}(A) \cup \text{Fr}(A)$;
4. $X = \text{Int}(A) \cup \text{Fr}(A) \cup \text{Int}(\mathbf{C}A)$ is a pairwise disjoint union.

Proof. (1) $\text{Fr}(A) = \overline{\mathbf{C}A} \cap \mathbf{C}[\overline{\mathbf{C}\mathbf{C}A}] = \overline{A} - \text{Int}(A)$. The proofs of the remaining assertions are entirely similar to this. □

Definition 23.6. $D \subset X$ is dense in X if $\overline{D} = X$.

Clearly, X is dense in X, and in fact X is the only *closed* set dense in X.

Proposition 23.7. The following four statements are equivalent:

1. D is dense in X.
2. If F is any closed set and $D \subset F$, then $F = X$.
3. Each nonempty basic open set in X contains an element of D.
4. The complement of D has empty interior.

Proof of this proposition is left as a homework.

24 Induced Topology

Let $Y \subset X$. if X carries a topology, we will define a topology for the *set* Y, called the *induced* (or relative) topology on Y. Its importance lies in this: To determine what any concept defined for topological spaces becomes when discussion is restricted to $Y \subset X$, we simply regard Y as a space with the *induced* topology and carry over the discussion *verbatim*.

Definition 24.1. Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$. The induced topology $\mathscr{T}_Y$ on Y is $\{Y \cap U \mid U \in \mathscr{T}\}$. $(Y, \mathscr{T}_Y)$ is called a *subspace* of $(X, \mathscr{T})$.

To verify $\mathscr{T}_Y$ is actually a topology on Y is trivial (if not see "Topology" by J.R. Mankres p.89). Let Y be a subspace of X, and $A \subset Y$. Since $(Y, \mathscr{T}_Y)$ is a space, we can form the closure of A, using $\mathscr{T}_Y$, to obtain $\overline{A_Y}$; but also $A \subset X$, so we can form $\overline{A}$, using $\mathscr{T}$. We now determine the relation between $\overline{A}$ and $\overline{A_Y}$, as well as that for the other elementary operations.

Theorem 24.1. *Let $(X, \mathscr{T})$ be a topological space, and $(Y, \mathscr{T}_Y)$ a subspace. Then:*

1. *If $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ is a basis (subbasis) for $\mathscr{T}$, $\{Y \cap U_\alpha \mid \alpha \in \mathscr{A}\}$ is a basis (subbasis) for $\mathscr{T}_Y$.*
2. *Let $A \subset Y$. Then A is $\mathscr{T}_Y$-closed if and only if $A = Y \cap F$, where F is $\mathscr{T}$-closed (that is, the closed sets in Y are the intersections of Y with sets closed in X).*
3. *$\overline{A_Y} = Y \supset \overline{A}$; $\overline{A_Y}' = Y \cap A'$; $Int_Y(A) = Y \cap Int(A)$; $Fr_Y(A) = Y \cap Fr(A)$.*

Proof.

1. (1) is trivial.
2. Let A be closed in Y; then $A = Y - W$, where W is open in Y, and since $W = Y \cap V$ where $V \in \mathscr{T}$, we find $A = Y - Y \cap V = Y \cap \mathbf{C}V$. Conversely, if $A = Y \cap F$, F closed in X, then $Y - A = Y \cap \mathbf{C}F$, showing that A is closed in Y.
3. $y \in \overline{A} \cap Y \Rightarrow \forall U(y) : U(y) \cap A \neq \emptyset$, and since $A \subset Y$, it follows that $\forall U(y) : (Y \cap U(y)) \cap A \neq \emptyset$ which shows that $y \in \overline{A_Y}$; the implications all reverse. the remaining statements are proved similarly.

□

Sets open in a subspace not need be open in the entire space; the following theorem gives a simple but useful case where this cannot occur.

Theorem 24.2. *Let Y be a subspace of X. If $A \subset Y$ is closed (open) in Y, and Y is closed (open) in X, then A is closed (open) in X.*

Proof. For $A = Y \cap K$, and since Y and K are both closed (open) in X, so also is the intersection. □