MA651 Topology. Lecture 4. Topological spaces 2

This text is based on the following books:

- *"Linear Algebra and Analysis" by Marc Zamansky*
- *"Topology" by James Dugundgji*
- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

25 Continuous Maps

We have been considering topologies on one given set; we now want to relate different topological spaces. Given $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, note that the a map $f : X \to Y$ relates the sets and also induces two maps $\hat{f} : \mathscr{P}(X) \to \mathscr{P}(Y)$, $\hat{f}^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$. Of these, $\hat{f}^{-1}$ should be used to relate the topologies, since it is the only one that preserves the Boolean operations involved in the definition of a topology. Thus the suitable maps $f : X \to Y$ are those for which simultaneously $\hat{f}^{-1} : \mathscr{T}_Y \to \mathscr{P}(X)$. Formally stated,

Definition 25.1. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each set open in Y is open in X (that is $\hat{f}^{-1}$ maps $\mathscr{T}_Y$ into $\mathscr{T}_X$).

Example 25.1. *A constant map $f : X \to Y$ is always continuous: The inverse image of any set U open in Y is either $\emptyset$ or X, which are open.*

Example 25.2. *Let X be any set, $\mathscr{T}_1$, $\mathscr{T}_2$ two topologies on X. The* bijective *map $1 : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if and only if $\mathscr{T}_2 \subset \mathscr{T}_1$. Note that a continuous map need not send open sets to open sets, and also that increasing the topology $\mathscr{T}_1$ preserves continuity.*

Example 25.3. *A map sending open sets to sets is called an* open map. *An open map need not be continuous.* $1 : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ *is open if and only if* $\mathscr{T}_1 \subset \mathscr{T}_2$, *but is not continuous whenever* $\mathscr{T}_1 \neq \mathscr{T}_2$.

Example 25.4. *Let* $Y \subset X$. *The relative topology* $\mathscr{T}_Y$ *can be characterized as the* smallest *topology on* Y *for which the inclusion map* $i : Y \to X$ *is continuous. For, if* $U \in \mathscr{T}$, *the continuity of* i *requires* $i^{-1}(U) = U \cap Y$ *to be open in* Y, *so that any topology for which* i *is continuous must contain* $\mathscr{T}_Y$.

The elementary properties are:

Proposition 25.1.

1. (Composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, so also, is $g \circ f : X \to Z$.
2. (Restriction of domain) If $f : X \to Y$ is continuous and $A \subset X$ is taken with a subspace topology, then $f|A : A \to Y$ is continuous.
3. (Restriction of range) If $f : X \to Y$ is continuous and $f(X)$ is taken with the subspace topology, then $f : X \to f(X)$ is continuous

Proof is left as a homework.
The basic theorem on continuity is:

Theorem 25.1. *Let* X, Y *be topological spaces, and* $f : X \to Y$ *a map. The following statements are equivalent:*

1. f *is continuous.*
2. *The inverse image of each closed set in* Y *is closed in* X.
3. *The inverse image of each member of a subbasis (basis) for* Y *is open in* X *(not necessarily a member of a subbasis, or basis for* X*!).*
4. *For each* $x \in X$ *and each neighborhood* $W(f(x))$ *in* Y, *there exists a neighborhood* $V(x)$ *in* X *such that* $f(V(x)) \subset W(f(x))$.
5. $f(\overline{A}) \subset \overline{f(A)}$ *for every* $A \subset X$.
6. $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ *for every* $B \subset Y$.

Proof.

- (1) $\Leftrightarrow$ (2), since $f^{-1}(Y - E) = X - f^{-1}(E)$ for any $E \subset X$.

- (1) $\Leftrightarrow$ (3). Let $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ be a subbasis for Y. If f is continuous, each $f^{-1}(U_\alpha)$ is open. Conversely, if each $f^{-1}(U_\alpha)$ is open, then because any open $U \subset Y$ can be written

$$U = \bigcup \{U_{\alpha_1} \cap \cdots \cap U_{\alpha_n} \mid \{\alpha_1, \cdots, \alpha_n\} \subset \mathscr{A}\},$$

we have that

$$f^{-1}(U) = \bigcup \{f^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}(U_{\alpha_n})\}$$

is a union of open sets and so is open.

- (1) $\Leftrightarrow$ (4). Since $f^{-1}(W(x))$ is open, we can use it for $V(x)$.

- (4) $\Leftrightarrow$ (5). Let $A \subset X$ and $b \in \overline{A}$; we show $f(b) \in \overline{f(A)}$ by proving each $W(f(b))$ intersects $f(A)$. For, finding $V(b)$ with $f(V(b)) \subset W(f(b))$,

$$\begin{aligned} b \in \overline{A} &\Rightarrow \emptyset \neq V(b) \cap A \\ &\Rightarrow \emptyset \neq f(V(b) \cap A) \subset f(V(b)) \cap f(A) \subset W(f(b)) \cap f(A). \end{aligned}$$

- (5) $\Leftrightarrow$ (6). Let $A = f^{-1}(B)$; then $f(\overline{A}) \subset \overline{f(A)} = \overline{f[f^{-1}(B)]} = \overline{B \cap f(X)} \subset \overline{B}$, so that $\overline{A} \subset f^{-1}\overline{B}$, as required.

- (6) $\Leftrightarrow$ (2). Let $B \subset Y$ be closed; then $\overline{f^{-1}(B)} \subset f^{-1}(B)$, and since always $f^{-1}(B) \subset \overline{f^{-1}(B)}$ (by Proposition 23.2 (a): for every set A: $A \subset \overline{A}$), this shows that $f^{-1}(B)$ is closed (by Proposition 23.1).

□

The formulation (4) of Theorem (25.1) shows that continuity is a "local" matter, a fact having many applications. Precisely,

Definition 25.2. An $f : X \to Y$ is *continuous at* $x_0 \in X$ if for each neighborhood $W(f(x_0))$ in Y, there exists a neighborhood $V(x_0)$ in X such that $f(V(x_0)) \subset W(f(x_0))$ (i.e. Theorem (25.1) (4) is satisfied at x_0).

From this viewpoint, the equivalence of (1) and (4) in Theorem (25.1) asserts: f is continuous according to Definition (25.1), if and only if it is continuous *at each point of* X.

26 Open Maps and Closed Maps

Definition 26.1. A map $f : X \to Y$ is called open (closed) if the image of each set open (closed) in X is open (closed) in Y.

We have already seen (Example (25.1) that a continuous map need not be an open map, and (Example (25.3)) that an open map need not be continuous. The following example shows that, in general, an open map need not be a closed map (even though it is continuous); the concepts "open map", "closed map", and "continuous map" are therefore independent.

Example 26.1. *Let $A \subset X$ and let $i : A \to X$ be the inclusion map $a \to a$. By Example (25.4) i is continuous. Furthermore, i is open (closed) if and only if A is open (closed) in X. Proof for "open": If A is open, and $U \subset A$ open in A, then by Theorem (24.2) (Let Y be a subspace of X. If $A \subset Y$ is closed (open) in Y, and Y is closed (open) in X, then A is closed (open) in X.), $i(U) = U$ is open in X. The proof for "closed" is analogues.*

Example 26.2. *If $f : X \to Y$ is* bijective, *then the conditions "f closed" and "f open are in fact equivalent. For, if f is open and $A \subset X$ is closed, then $A = X - U$ and $f(A) = f(X) - f(U) = Y - f(U)$, so $f(A)$ is also closed. As Examples (25.2) and (25.3) show, "bijective open" and "bijective continuous" are still distinct notions.*

The behavior of inverse images further emphasizes the distinction between open maps and closed maps:

Theorem 26.1.

1. *Let $p : X \to Y$ be a closed map. Given any subset $S \subset Y$ and any open U containing $p^{-1}(S)$, there exists an open $V \supset S$ such that $p^{-1}(V) \subset U$.*
2. *Let $p : X \to Y$ be an open map. Given any subset $S \subset Y$, and any closed A containing $p^{-1}(S)$, there exists a closed $B \supset S$ such that $p^{-1}(B) \subset A$.*

Proof. We prove only (1) since the proof of (2) is similar. Let $V = Y - p(X - U)$; since $p^{-1}(S) \subset U$, it follows that $S \subset V$, and because p is closed, V is open in Y. Observing that

$$p^{-1}(V) = X - p^{-1}[p(X - U)] \supset X - [X - U] = U$$

completes the proof. □

Theorem (26.1) is particularly important and has significant consequences; its most frequently occurring form is with S a single point.
We now give some characterization of open maps and of closed maps.

Theorem 26.2. *The following fore properties of a map $f : X \to Y$ are equivalent:*

1. *f is an open map.*
2. *$f[Int(A)] \subset Int[f(A)]$ for each $A \subset X$.*

3. *f sends each member of a basis for X to an open set in Y.*

4. *For each $x \in X$ and neighborhood U of x, there exists a neighborhood W in Y such that $f(x) \in W \subset f(U)$.*

Proof.

- (1) $\Leftrightarrow$ (2). Since $\text{Int}(A) \subset A$, we have $f[\text{Int}(A) \subset f(A)$. By hypothesis, $f[\text{Int}(A)]$ is open, and because $\text{Int}[f(A)]$ is the largest open set in $f(A)$, we must have $f[\text{Int}(A)] \subset \text{Int}[f(A)]$.

- (2) $\Leftrightarrow$ (3). Let U be a member of a basis. Being open $U = \text{Int}(U)$ and so $f(U) = f[\text{Int}(U)] \subset \text{Int}[f(U)] \subset f(U)$; thus, $f(U) = \text{Int}[f(U)]$ and therefore $f(U)$ is open.

- (3) $\Leftrightarrow$ (4). Given x and neighborhood U of x, find a member V of the basis for X such that $x \in V \subset U$ (by Theorem (21.2)) and let $W = f(V)$.

- (4) $\Leftrightarrow$ (1). Let U be open in X. by hypothesis, each $y \in f(U)$ has a neighborhood $W(y) \subset f(U)$ so that $f(U) = \cup\{W(y) \mid y \in f(U)\}$ shows that $f(U)$ is open.

□

Theorem 26.3. *$p : X \to Y$ is a closed map if and only if $\overline{p(A)} \subset p(\overline{A})$ for each set $A \subset X$*

Proof. If p is closed, then by Proposition (23.1), $p(\overline{A})$ is closed; since $p(A) \subset p(\overline{A})$, we obtain $\overline{p(A)} \subset \overline{p(\overline{A})} = p(\overline{A})$ as required. Conversely, if the condition holds and A is closed, then $p(A) \subset \overline{p(A)} \subset p(\overline{A}) = p(A)$ shows that $\overline{p(A)} \subset p(A)$, so that $p(A)$ is closed. □

27 Homeomorphism

Definition 27.1. A continuous bijective map $f : X \to Y$, such that $f^{-1} : Y \to X$ is also continuous is called a homeomorphism (or a bicontinuous bijection) and denoted by $f : X \cong Y$. Two spaces X, Y are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f : X \cong Y$.

Example 27.1. *The map $x \to \frac{x}{1+|x|}$ is a homeomorphism of R and $]-1, +1[$. Interpreting x as a vector in R^n, this map shows that R^n is homeomorphic to its unit ball $B(0; 1)$.*

Example 27.2. *The extendended real line $\tilde{R}$ is homeomorphic to $[-1, +1]$, since the map $x \to \frac{x}{1+|x|} \mid (x \in R)$, $\pm\infty \to \pm 1$ is a homeomorphism.*

Example 27.3. *Let $p = (0, 0, 1)$ be the north pole of the sphere S^2; then $S^2 - \{p\} \cong R^2$, since the stereographic projection from p, which sends*

$$(x_1, x_2, x_3) \in S^2 - \{p\} \text{ to } (\tfrac{x_1}{1-x_3}, \tfrac{x_2}{1-x_3}, 0) \in R^2 \subset R^3$$

is easily verified to be a homeomorphism. This is the familiar process in complex analysis, which completes the complex numbers (geometrically, R^2) by adding a "point of infinity" to get S^2. in similar fashion, we have $S^n - \{(0, \cdots, 0, 1)\} \cong R^n$.

The importance of homeomorphisms results from the observation that a homeomorphism is also an open map; for, it then follows at once that a homeomorphism $f : X \cong Y$ provides *simultaneously* a bijection for the underlying spaces *and* for the topologies: that is, both $f : X \to Y$ and the induced $\hat{f}|\mathscr{T}(X) : \mathscr{T}(X) \to \mathscr{T}(Y)$ are bijective. Then their significance in this: Any assertion about X *as a topological space* is also valid for each homeomorphism of X; more precisely, every property of X expressed entirely in terms of set operations and open sets (that is, any topological property of X) is also possessed by each space homeomorphic to X.

Somewhat more generally, we call any property of spaces a *topological invariant* if whenever it is true for one space X it is also true for every space homeomorphic to X; trivial examples are cardinal of point set, and cardinal of topology. With this terminology, every topological property of a space is a topological invariant, homeomorphic spaces have the same topological invariants, and Topology can be described as the study of topological invariants.

This description of topology can be expressed more formally: Observe that homeomorphism is an equivalence relation in the class of all topological spaces, since

(a) $1 : X \cong X$

(b) $[f : X \cong Y] \Rightarrow [f^{-1} : Y \cong X]$

(c) $[f : X \cong Y] \wedge [g : Y \cong Z] \Rightarrow [g \circ f : X \cong Z]$

that is easy to verify. Consequently, the relation of homeomorphism decomposes the class of all topological spaces into mutually exclusive classes, called *homeomorphism types.* In these terms, Topology studies invariants of homeomorphism types.

Homeomorphism frequently allows the reduction of a given problem to a simpler one: A space that is given, or constructed, in some complicated manner may possible be shown homeomorphic to something more familiar, and its topological properties thereby more easily determined. For example, it is known that the Riemann surface of an algebraic functions is homeomorphic to a sphere S^2 having suitably many attached handles. Unfortunately, to show that two given spaces are homeomorphic is usually difficult, with construction of a homeomorphism being the only general method. In some special cases, such as two-dimensional manifolds, other (algebraic) techniques have been devised.

It is frequently important to know that two spaces are *not* homeomorphic, as, for example, R and R^2. This problem is somewhat simpler than the former; it is generally solved by displaying a topological invariant possessed by only one of spaces. Topological invariants not possessed by all spaces are therefore important, in this course we will see many such invariants.

Theorem 27.1. *Let $f : X \to Y$ be* bijective. *The following properties of f are equivalent:*

1. *f is a homeomorphism.*
2. *f is continuous and open.*
3. *f is continuous and closed.*
4. *$f(\overline{A}) = \overline{f(A)}$ for each $A \subset X$*

Proof.

- (1) $\Leftrightarrow$ (2). The requirement that the map $f^{-1} : Y \to X$ be continuous is equivalent to the stipulation that for each open $U \subset X$, the set $(f^{-1})^{-1}(U) = f(U)$ be open in Y.
- (2) $\Leftrightarrow$ (3). This is equivalent to Example (26.2).
- (3) $\Leftrightarrow$ (4). Continuity of f yields $f(\overline{A}) \subset \overline{f(A)}$, and because f is closed, Theorem (26.3) shows that also $\overline{f(A)} \subset f(\overline{A})$.

□

One frequently used technique for establishing that a given $f : X \to Y$ is a homeomorphism is simply to exhibit a continuous $g : Y \to X$ in accordance with

Theorem 27.2. *Let $f : X \to Y$ and $g : Y \to X$ be continuous and such that both $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then f is a homeomorphism, and in fact, $g = f^{-1}$.*

Proof. We know (see Proposition (7.3): "Let $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = 1_X$. then f is injective and g is surjective.") that both f and g are bijective, and it is trivial to see that $g = f^{-1}$; since both f and g are continuous, the proof is complete. □

For subspaces,

Theorem 27.3. *Let $f : X \cong Y$ and $A \subset X$. Then $f|A : A \cong f(A)$ and $f|X - A : X - A \cong Y - f(A)$.*

Proof. Let $g = f^{-1}|f(A)$; then g is continuous, by Proposition (25.1) (2), and the pair of maps $f|A$,g satisfies Theorem (27.2). The second part is proved in the same way. □

Definition 27.2. If Z is any space and $f : X \to Z$ is a map establishing $X \cong f(X) \subset Z$, then f is called an *embedding map* of X into Z.

28 Continuity from a "local" viewpoint.

We would like to bring another definition of continuity using limits. But first we should consider limit points in topological spaces using concept of a filter. Before we introduce abstract definitions let us briefly discuss concepts of limit and convergence which are associated with sequences of real numbers.

The expression "a sequence (x_n) of real numbers has a limit (or converges to) a real number x_0" means "every open interval containing x_0 contains all but a finite number of the x_n". Then we say that x_n converges, or tends to, or has a limit x_0 when n tends to infinity.

For real-valued functions we also define the expression "$f(x)$ tends to y when x tends to x_0" or "... when x tends to zero on the right", etc.

Other elementary concepts of limits are also important. For example a point x_0 is a point of accumulation of a countable set if every open set containing x_0 contains a point of the set other than x_0. We have also the idea of a subsequence extracted from a given sequence, the Bolzano-Weierstrass theorem on R: from every infinite bounded sequence we can extract a convergent sequence; and the concept of a doubled sequence $(x_{p,q})$ which converges to x_0 when p and q tend to infinity.

If we consider the case of a sequence (x_n) tending to x_0 when n tends to infinity we can make the following observations about the definition:

1. The expression "all the x_n except for a finite number" means that we consider the complements (with respect to the set of the x_n) of finite subsets. If we denote these complements by A, A', ... none of them is empty and the intersection $A \cap A'$ of any two is again the complement of a finite subset. Thus the set of complements with respect to the set of x_n, of the finite subsets is a *fundamental family* which does not contain the empty set.

2. The expression "every open interval X containing x_0 contains all the x_n except for a finite number" means that every $X \in \mathscr{B}(x_0)$ contains in A.

These examples give rise to the definitions which follow.

28.1 The concept of a filter

28.1.1 Definition of a filter

Definition 28.1. A non-empty family of subsets of E is called a filter on E if it is a fundamental family which does not contain $\emptyset$.

Filters will be denoted by $\mathscr{F}$, $\mathscr{F}'$ etc.
If a fundamental family does not contain Ø the intersection of a finite number of its members is non-empty (see observation 2 after Definition (19.1)). Thus every finite intersection of sets of a filter is non-empty.

Example 28.1. *In a topological space, the basis of open neighborhoods $\mathscr{B}(x)$ of each point x is a filter. On the other hand, a basis $\mathscr{T}$ for the topology is not a filter since $\emptyset \in \mathscr{T}$.*

Example 28.2. *On the real line, a set non-empty open intervals all of which have the same left-hand end point (or the same right-hand end point) is a filter.*

Example 28.3. *Let N be the set of natural numbers. For each n let X be the set of integers greater than n. The family consisting of these sets X forms a filter since none of them is empty, and is X and X' are the two sets defined by n and n', respectively, $X \cap X'$ is the set of integers greater than $max(n, n')$ and so belongs to the family.*

28.1.2 Comparison of filters

If $\mathscr{F}$ and $\mathscr{F}'$ are two filters on the same set E, $\mathscr{F}$ is said to be *finer* than $\mathscr{F}'$, or $\mathscr{F}'$ is *coarse* than $\mathscr{F}$, if the fundamental family $\mathscr{F}$ is finer than $\mathscr{F}'$ (see Definition (19.2)), i.e. if every $A' \in \mathscr{F}'$ contains an $A \in \mathscr{F}$.

If $\mathscr{F}$ is finer than $\mathscr{F}'$ and $\mathscr{F}'$ is finer than $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}'$ are said to be *equivalent.*

Example 28.4. *Suppose we are given bases $\mathscr{T}$ and $\mathscr{T}'$ for two topologies on a set E. If $\mathscr{T}'$ is coarse than $\mathscr{T}$, then for every $x \in E$ the filter $\mathscr{B}_{\mathscr{T}'}(x)$ is coarse than $\mathscr{B}_{\mathscr{T}}(x)$. If $\mathscr{T}$ and $\mathscr{T}'$ are equivalent, $\mathscr{B}_{\mathscr{T}'}(x)$ and $\mathscr{B}_{\mathscr{T}}(x)$ are equivalent.*

Let $\mathscr{F}$ be a filter on a set E. A filter obtained by taking a subset of each set of $\mathscr{F}$ is called a *filter extracted from $\mathscr{F}$*. A filter extracted from the filter $\mathscr{F}$ is finer than $\mathscr{F}$.

Example 28.5. *Let $E = N$ and let $\mathscr{F}$ be the filter of complements of finite subsets (the natural filter on N or Fréchet filter, see below). Consider an infinite sequence of integers n_k. Let $\mathscr{F}$ be the filter of complements of finite subsets of the set of n_k. Every element of $\mathscr{F}$ contains an element of $\mathscr{F}'$.*

Let $\mathscr{F}$ be a filter on a set E, X a non-empty subset of E. If every $A \in \mathscr{F}$ meets X, the set $A \cap X$, where $A \in \mathscr{F}$, is called the *filter induced on X by $\mathscr{F}$*.

Example 28.6. *It is clear that the filter $\mathscr{F}'$ in Example (28.5) can be considered as the filter induced by the natural filter $\mathscr{F}$ on the set of integers (n_k).*

28.1.3 Fréchet filter, filter of sections

On the set of integers N consider the filter $\mathscr{F}$ consisting of sets A of integers $m \geq n$ (for arbitrary n). Thus every $A \in \mathscr{F}$ is the complement of a finite subset of N.

Let $\mathscr{F}'$ be the set of complements of finite subsets of N. $\mathscr{F}'$ is clearly a filter. If $A' \in \mathscr{F}'$ there exists a finite subset ϕ' of N such that $A' = \mathbf{C}\phi'$. If n' is the greatest integer contained in ϕ' and A the set of $m \geq n'$, then $A' \supset A$. Conversely, if $A \in \mathscr{F}$ we have $A \in \mathscr{F}'$, so that the filters $\mathscr{F}$ and $\mathscr{F}'$ are equivalent. We have here made use of the total order on N

Using the definition of the filter $\mathscr{F}$ above we can define a filter on an *arbitrary* (not necessary totally) *ordered set* E.

Let E be a set which we shall suppose ordered by relation $\geq$. For every $x \in E$ let A be the set of $y \in E$ such that $y \geq x$. A is a subset of E called the *section* corresponding to x. Let $\mathscr{F}$ be the set of A. No A is empty, and if A and A' are two elements of $\mathscr{F}$ defined by x and x' respectively, their intersection is the empty set of $y \in E$ such that $y \geq x$ and $y \geq x'$. However this set may be empty.

We therefore introduce a further hypothesis which will ensure that $A \cap A'$ is not empty. We suppose that A is ordered and that if x, $x' \in E$ there exists y such that $y \geq x$ and $y \geq x'$. The family $\mathscr{F}$ we have just defined then becomes a filter. When E is the set of integers N with its usual order we have the set defined above.

In particular, let Φ be the family of finite subsets of an arbitrary set E. We order Φ by inclusion. If $\phi, \phi' \in \Phi$, since $\phi \cup \phi' \in \Phi$, $\phi \subset \phi \cup \phi'$, $\phi' \subset \phi \cup \phi'$, the set Φ has the properties required above. It follows that the family of subsets of E containing an element ϕ of Φ is a filter.

Definition 28.2. The natural filter on a set E is the filter consisting of the complements of finite subsets of E.

Definition 28.3. The filter of sections on the set Φ of finite subsets of a set E is the filter whose generic element is the set of finite subsets of E containing a given finite subset.

This filter is also called the *filter of sections* associated with E. It is important to remember that the filter is defined on Φ, the family of finite subsets of E, even thought we call it the filter of sections of E.
However, if $E = N$, an element of the filter of sections can be identified with the complement of a finite subset and so we can then regard the natural filter (Fréchet filter) and the filter of sections as equivalent.

28.1.4 Images of a filter

Proportions (19.1) and (19.3) show that if f is a mapping of a set E into a set E':

1. The image by f of a filter on E is a filter on E',
2. The inverse image by f of a filter $\mathscr{F}'$ on E' is a filter on E if every set of $\mathscr{F}'$ meets $f(E)$.

In particular, if f is a mapping of E onto E' the direct and inverse images of filters are again filters.

Example 28.7. *Let (x_n) be a sequence of points in a set E, i.e. a mapping of N into E. The image in E under this mapping of the natural filter on N is a filter, but, in general, it does not consist of the family of complements of finite subsets of the set of values of the sequence. If for example E consist of a single element a, we have $x_n = a$ for all n, and the image of the natural filter is a filter consisting of a single element E, whilst the complements of a finite subset is $\emptyset$*

28.2 Limits in topological spaces

28.2.1 Limit point of a filter

Definition 28.4. Let $(E, \mathscr{T}$ be a topological space and $\mathscr{F}$ a filter on E. We say that a point $x \in E$ is a limit or limit point of $\mathscr{F}$ if $\mathscr{F}$ is finer than the filter $\mathscr{B}(x)$ (the basis of open neighborhoods), i.e. if every $X \in \mathscr{B}(x)$ contains an $A \in \mathscr{F}$.

We then say that $\mathscr{F}$ converges to x or has x as a limit, or that $\mathscr{F}$ converges or is convergent (if we do not need to specify the limit).
Remarks:

- A filter does not necessary converge. For example the natural filter on N with the discrete topology.
- A filter may have more than one limit point.
- If x is a limit point of $\mathscr{F}$, x may belong to some $A \in \mathscr{F}$ or may not belong to any $A \in \mathscr{F}$. For example the filter $\mathscr{B}(x)$ consisting of the open neighborhoods of x converges to x, and x belongs to every $X \in \mathscr{B}(x)$. On the other hand, the set of open intervals of R all having the same left-hand end point x is a filter which converges to x, but x does not belong to any member of the filter.

28.2.2 Point adherent to a filter

Definition 28.5. Let $(E, \mathscr{T}$ be a topological space and $\mathscr{F}$ a filter on E. A point $x \in E$ is said to be adherent to $\mathscr{F}$ if it is adherent to every $A \in \mathscr{F}$, i.e. if every $X \in \mathscr{B}(x)$ meets every $A \in \mathscr{F}$

Example 28.8. *Consider in R the set of points $1/n$, $1 - 1/n$ (where $n \in N$) and the point 2. Let $\mathscr{F}$ be the filter consisting of the complements of finite subsets of this set. $\mathscr{F}$ has the points 0 and 1 as adherent points.*

28.2.3 Relation between limit points and adherent points

Let $\mathscr{F}$ be a filter on a topological space E. Suppose that x is a limit of $\mathscr{F}$. Then every $X \in \mathscr{B}(x)$ contains an $A \in \mathscr{F}$. Now if A' is an arbitrary element of $\mathscr{F}$, $A \cap A'$ is non-empty, so that X meets every set of $\mathscr{F}$. Thus:

Proposition 28.1. Every limit point is an adherent point.

Now suppose that x is adherent point to $\mathscr{F}$. Consider the family of subsets $A \cap X$ of E, where A is an arbitrary element of $\mathscr{F}$, X an arbitrary element of $\mathscr{B}(x)$. None of these sets is empty since x is adherent to $\mathscr{F}$.
If we consider two of them we have

$$(A \cap X) \cap (A' \cap X') = (A \cap A') \cap (X \cap X').$$

Since $A \cap A'$ contains an $A'' \in \mathscr{F}$ and $X \cap X'$ contains $X'' \in \mathscr{B}(x)$

$$(A \cap X) \cap (A' \cap X') \supset A'' \cap X''.$$

Thus the family of $A \cap X$ is a filter $\mathscr{F}'$, and since $A \cap X \subset A$, $\mathscr{F}'$ is extracted from $\mathscr{F}$.

Finally, every $X \in \mathscr{B}(x)$ contains $A \cap X$, an element of $\mathscr{F}'$, so that $\mathscr{F}'$ converges to x. Thus when x is adherent to a filter $\mathscr{F}$, there is a filter $\mathscr{F}'$ extracted from $\mathscr{F}$ and converging to x.

Conversely, suppose that given a filter $\mathscr{F}$ there is a filter $\mathscr{F}'$ extracted from $\mathscr{F}$ and converging to a point x. Then every $X \in \mathscr{B}(x)$ contains an $A' \in \mathscr{F}'$ and meets every element of $\mathscr{F}'$. Now by definition of an extracted filter we obtain $\mathscr{F}$ by taking a subset of *every* set of $\mathscr{F}$, so that every $X \in \mathscr{B}(x)$ meets every $A \in \mathscr{F}$. Hence:

Theorem 28.1. *A point x is adherent to a filter $\mathscr{F}$ if and only if there is a filter $\mathscr{F}'$ extracted from $\mathscr{F}$ and converging to x.*

28.3 Images of limits, sequences

Let f be a mapping of a set E into a set E'. If we are given a filter $\mathscr{F}$ on E we have seen that $f(\mathscr{F})$ is a filter on E'. If $A \in \mathscr{F}$, the set of $f(A)$ forms a filter on E', and to give meaning to the expression: the filter $f(\mathscr{F})$ converges to a point $x_0' \in E'$, we must give a topology on E'. We shall now consider the cases with E' is a topological space and then where E and E' are both topological spaces.

28.3.1 The case of a function with values in a topological space.

Let E be a set, E' a topological space defined by a basis $\mathscr{T}'$, f a mapping of E into E', and $\mathscr{F}$ a filter on E.

Definition 28.6. We say that $f(x)$ converges to a point $x_0' \in E'$ along $\mathscr{F}$ if the filter $f(\mathscr{F})$ converges to x_0' in the space E'.

We also say: f has limit x_0' along $\mathscr{F}$, that x_0' is a limit of f along $\mathscr{F}$, or that x_0' is a *limit value* of f.
The filter $f(\mathscr{F})$ may have no limit point, but one or more adherent points in E'. Such points are called *adherent values.*
Finally, the filter $f(\mathscr{F})$ may have neither limit points nor adherent points.
The definition of a limit point of a filter shows that it comes to the same thing to say:

Definition 28.7. $f(x)$ converges to a point $x_0' \in E'$ along $\mathscr{F}$ if every $X' \in \mathscr{B}_{\mathscr{T}'}(x_0')$, $f^{-1}(X')$ contains an $A \in \mathscr{T}$.

Sequences provide the most important example of these definitions.

28.3.2 Convergence of sequences

The family of complements of finite subset of N is a filter which we have called the natural filter. A sequence in a topological space E is a mapping f of N into E whose value $f(n)$ (for $n \in N$) is written x_n. Instead of "x_n tends to x_0 (or converges to x_0) along the natural filter" we usually say "x_n tends to x_0 when n tends to infinity" and write

$$x_0 = \lim_{n \to \infty} x_n$$

We sometimes abbreviate further by saying "x_n tends to or converges to x_0" or "has the limit x_0". It is understood that the convergence is along the natural filter.
Remarks:

1. When we say that x_n tends to x_0 we mean that every $X \in \mathscr{B}(x_0)$ contains an $f(A)$, which, here, is the set of points x_k where k belongs to the complement of a finite subset of N.

Consequently for every $X \in \mathscr{B}(x_0)$ there is an integer $p(X)$ such that for every $k \geq p(X)$ we have $x_k \in X$. Conversely, if for every X there exits an integer $p(X)$ such that $x_k \in X$ for $k \geq p(X)$ we see that every $X \in \mathscr{B}(x_0)$ contains all the x_k for which k belongs to the complement of a finite subset of N. This in fact amounts to the elementary definition of convergence of x_n to x_0.

2. If A is the complement of a finite subset of N, an element, that is, of the natural filter, the set of *points* x_n where $n \in A$ is not, in general, the complement of a finite set of points of the sequence.

3. If the points or values x_n are all distinct, and if e denotes the set of points x_n, the image of the natural filter by the sequence (x_n) consists of the complements of finite subsets of e.

4. Let e denotes the set of points x_n. If x_0 is adherent to e, x_0 is not necessarily an adherent point of the filter. But if x_0 is an adherent point of the sequence (x_n), x_0 is adherent to e. For example, on R, the sequence $(1/n)$ has the single adherent point 0, but every point $1/n$ is adherent to the set of values of the sequence.

5. Every subsequence of a sequence converging to x_0 also converges to x_0.

28.3.3 The case of a mapping of a topological space into a topological space.

Let $(E, \mathscr{T})$ and $(E', \mathscr{T}')$ be two topological spaces, f a mapping of E into E', and $\mathscr{F}$ a filter on E. The convergence of $f(\mathscr{F})$ in E' is unrealistic to the convergence of $\mathscr{F}$ in E. This is illustrated by the example of convergent sequence, for which $\mathscr{F}$ is the natural filter and we may consider N as a subset of the topological space R.
However, if $\mathscr{F}$ converges in E to a point $a \in E$ (either if this just happens to be so or if we have chosen $\mathscr{F}$ so as to converge to a) we may then say "$f(x)$ tends to x'_0 when x tends to a along $\mathscr{F}'$ which means the same as "$\mathscr{F}$ converges to a in E and $f(\mathscr{F})$ to x'_0 in E', simultaneously".
We can express this definition in terms of the bases of neighborhoods $\mathscr{B}_{\mathscr{T}}(a)$, $\mathscr{B}_{\mathscr{T}'}(x'_0)$ and the elements A of $\mathscr{F}$. Thus to say that $f(\mathscr{F})$ converges to x'_0 means that *every* $X' \in \mathscr{B}_{\mathscr{T}'}(x'_0)$ contains an $f(A)$, or that $f^{-1}(X') \supset A$. We also say : for every $X' \in \mathscr{B}_{\mathscr{T}'}(x'_0)$ there is an $A \in \mathscr{F}$ such that $f(A) \subset X'$.
In the same way we define a limit x'_0 of f when x tends to a along $\mathscr{F}$.
Particular cases:

1. If the filter $\mathscr{F}$ is $\mathscr{B}_{\mathscr{T}}(a)$, the basis of open neighborhoods of a in E, which converges to a by definition, instead of : $f(x)$ tends to x'_0 when x tends to a along $\mathscr{B}_{\mathscr{T}}(a)$, we say: $f(x)$ tends to x'_0 when (or if) x tends to a, and we write:

$$x'_0 = \lim_{x \to a} f(x).$$

2. Let (x_n) be a sequence of points of E which converges to a. Here, $\mathscr{F}$ is the image, by the sequence, of the family of complements of finite subsets of N. Instead of: x_n tends to a along $\mathscr{F}$, we have agreed to say: x_n tends to infinity. What does it mean to say that $f(x_n)$ tends to x'_0 if x_n tends to a when n tends to infinity? For every $X' \in \mathscr{B}_{\mathscr{T}'}(x'_0)$ there exists $A \in \mathscr{T}$ such that $f(A) \subset X'$. Now an element A is the set of x_n corresponding to all but a finite number of the n. *There therefore exists an integer P such that for every $n \geq P$, $f(x_n) \in X'$.*

   This latter phrase expresses: $f(x_n)$ tends to x'_0 when x_n tends to a. We then write

$$x'_0 = \lim_{x \to \infty} f(x_n),\ (x_n \to a),$$

   instead of

$$x'_0 = \lim_{\substack{x \to a \\ \mathscr{F}}} f(x_n).$$

Example 28.9. *Let f be a real-valued function of the real variable x, defined on R. The topology of R is that defined by the open intervals.*

1. *To say that $f(x)$ tends to x'_0 when x tends to a means that for* every *open interval X' containing x'_0, there is an open interval X containing a and such that $f(X) \subset X'$, or, that* for every $\varepsilon > 0$ *there exists an α such that for every x satisfying $|x - a| < \alpha$, we have $|f(x) - x'_0| < \varepsilon$.*

2. *For a function of the same sort we take as a filter having a as limit the image $\mathscr{F}$ of the natural filter be a sequence (x_n). The image of $\mathscr{F}$ by f is the set of $f(x_k)$, where k takes all integral values except for a finite set. To say that $f(x_n)$ has a limit x'_0 when x_n tends to a along $\mathscr{F}$ means that* for every $\varepsilon > 0$ *there is an integer P such that if $n \geq P$, $|f(x) - x'_0| < \varepsilon$.*

3. *Again for the same function, we take as filter $\mathscr{F}$ having limit a, the family of open intervals $]a, \alpha[$ ($]\beta, a[$) all having left-hand end-points a, (right-hand end-points a). If $f(x)$ has a limit x'_0 when x tends to a along $\mathscr{F}$ we say that $f(x)$ tends to x'_0 when x tends to a on the right (on the left) and write*

$$x'_0 = \lim_{x \to a+} f(x),\ (= \lim_{x \to a-} f(x))f(x).$$

28.4 Local definition of a continuous map

Definition 28.8. Let f be a mapping of the topological space E into the topological space F. f is continuous at the point $a \in E$ if

$$\lim_{x \to a} f(x) = f(a)$$

We recall $\lim_{x \to a} f(x)$ is the limit along the filter $\mathscr{B}(a)$ which is the basis of open neighborhoods of a in E. The definition can also be expressed: f is continuous at the point $x \in E$ if $f(x)$ tends to $f(a) \in F$ when x tends to a along $\mathscr{B}(a)$.

Definition 28.9. Let f be a mapping of the topological space E into the topological space F. We say that f is continuous in, or on, E if it is continuous at every point of E.

Proposition 28.2. A mapping f of the topological space E into the topological space F is continuous at the point $a \in E$ if one of the following conditions is satisfied:

1. For every $Y \in \mathscr{B}(f(a))$ in F, $f^{-1}(Y)$ is a neighborhood of a in E, i.e. it contains an $X \in \mathscr{B}(a)$.

2. For every neighborhood W of $f(a)$ in F, $f^{-1}(Y)$ is a neighborhood of a in E.

3. For every filter $\mathscr{F}$ in E converging to a, $f(\mathscr{F})$ converges to $f(a)$ in F.

Proof of this proposition is left as a homework.