MA651 Topology. Lecture 5. Cartesian Product Topology. Connectedness.

This text is based on the following books:

- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Topology" by James Dugundgji*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

29 Cartesian Product Topology.

Let $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in \mathscr{A}\}$ be any family of topological spaces. We would like to topologize the Cartesian product $\prod\limits_{\alpha} X_\alpha$.

There are of course various topologies we can put on $\prod\limits_{\alpha} X_\alpha$, for example, the discrete, or indiscrete topologies. It turns out that these do not yield interesting theorems. All told, we have the following criteria to guide us in making a final choice:

1. The topology should be mathematically fruitful.
2. It should have some relations to the given topologies. That is, we should be able to draw conclusions about the product space from the coordinate spaces, and conversely.
3. We would like the projections $p_\beta : \prod\limits_{\alpha \in \mathscr{A}} X_\alpha \to X_\beta$ to be continuous.
4. In the case $\mathscr{A}$ consists of the integers $1, 2, \dots, n$ (or actually, any finite set), and each $X_i = R$, we would like the space $\prod\limits_{i=1}^{n} X_i$ to identify (in sense of homeomorphism) with the Euclidean space R^n.

While a satisfactory topology was easy for finite products, Tietze in 1923 was the first to topologize infinite products with any degree of success. He defined a topology on $\prod_{\alpha \in \mathscr{A}} X_\alpha$ be specifying the subbasis sets as those of the form $\prod_{\alpha \in \mathscr{A}} G_\alpha$, where G_α is open in X_α. This was a natural choice, since it was known to work very well for finite products. However, subsequent experience was shown Tychonov's 1930 definition to be the more useful one, and this is the one today is known as the product topology. In particular, Tychonov's Theorem which we will learn later (but see p.234 of Mankres's book) is considered by many the most important theorem in set topology, and it is true for Tychonov's, but not for Tietze's topology.

Definition 29.1. Let $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in \mathscr{A}\}$ be any family of topological spaces. The cartesian product topology in $\prod_\alpha X_\alpha$ is that having for subbasis all set $p_\beta^{-1}(U_\beta)$, where U_β ranges over all members of $\mathscr{T}_\beta$ and β over all elements of $\mathscr{A}$, i.e. the product topology generated by $\{p_\alpha^{-1}(U_\alpha) \mid \alpha \in \mathscr{A}$ and U_α is $\mathscr{T}_\alpha$-open$\}$

The basic open sets of the cartesian product topology look like $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$, where $\alpha_1, \ldots, \alpha_n$ are in $\mathscr{A}$ and U_{α_i} is open in X_{α_i} for $i = 1, 2, \ldots, n$. Since $p_\beta^{-1}(U_\beta) \cap p_\beta^{-1}(V_\beta) = p_\beta^{-1}(U_\beta \cap V_\beta)$ whenever $\beta \in \mathscr{A}$ and U_β, $V_\beta \in \mathscr{T}_\beta$, we may always assume for convenience that the α_i's are chosen to be distinct in the above expression.

Note that a set $\prod_{\alpha \in \mathscr{A}} G_\alpha$, where each G_α is open in each X_α and $G_\alpha \neq X_\alpha$, for *infinitely* many α is not open in $\prod_{\alpha \in \mathscr{A}} X_\alpha$ in the cartesian product topology. This is immediate since $\prod_{\alpha \in \mathscr{A}} G_\alpha$ can contain no basic open set $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ as a subset if each (or at least infinitely many) $G_\alpha \neq X_\alpha$. this makes it easy to see that in general, when $\mathscr{A}$ is infinite and infinitely many X_α have at least two points, then

$$\text{Tychonov product topology} \subsetneqq \text{Tietze product topology}$$

since the Tietze product topology is generated by

$$\{\prod_{\alpha \in \mathscr{A}} G_\alpha \mid G_\alpha \text{ is open in } X_\alpha \forall \alpha \in \mathscr{A}\}$$

In fact,

$$\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i}) = \prod_{\alpha \in \mathscr{A}} G_\alpha, \text{ where } G_\alpha = \begin{cases} U_\alpha, & \alpha = \alpha_i,\ i = 1, \ldots, n \\ X_\alpha, & \alpha \neq \alpha_i \end{cases}$$

so each basis Tychonov-open set is Tietze-open. When A is finite, the two topologies coincide.

Example 29.1. *Let $\mathscr{A} = Z^+$ and $X_i = R$ for each positive integer i. Then $X = \prod_{i \in \mathscr{A}} X_i = \prod_1^\infty X_i$ may be thought of as the set of all real-valued sequences. Let S consist of all sets $\prod_1^\infty A_i$, where A_i is open in R and $A_i = R$ for all but at most finitely many values of i. Then S generates a product topology on X.*
A subbasis open set is one of the form $p_i^{-1}(U_j)$, where U_j is Euclidean open in R. Since $p_j^{-1}(U_j) = \{x \mid x \in \prod_1^\infty X_i \text{ and } x_j \in U_j\}$, then $p_j^{-1}(U_j)$ consist of all $x : Z^+ \to R$ with $a < x_j < b$; x_i may be any real number for $i \neq j$.
A basis open set is a finite intersection of subbasic open sets, say $s = \bigcap_{i=1}^n p_{ji}(U_{ji})$, where $j_1, \ldots, j_n$ are positive integers and each U_{ji} is open in R. Then $x \in s$ exactly when $x : Z^+ \to R$ and $x_{ji} \in U_{ji}$ for $i = 1, \ldots, n$; for $i \neq j_1, \ldots, j_n$, x_i may be any real number.

When the index set is finite, say $\mathscr{A} = \{1, \ldots, n\}$ then the function $\prod_1^n X_i$ can be identified with the $n-$tuple set $X_1 \times X_2 \times \cdots \times X_n$ by thinking of x in $\prod_1^n X_i$ as $(x_1, \ldots, x_n)$. the identification is a topological one if we replace subbasis open sets $p_i^{-1}(U_i)$ in the product topology with subsets $X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$ of $X_1 \times \cdots \times X_n$.

Theorem 29.1. *Let $\mathscr{A} = \{1, \ldots, n\}$, where n is a positive integer. Let τ be the topology on $X_1 \times \cdots \times X_n$ generated by $\{G_1 \times \cdots \times G_n \mid G_i \in T_i,\ i = 1, \ldots, n\}$. Then,*

$$(\prod_{i=1}^n X_i, P) \cong (X_1 \times \cdots \times X_n, \tau)$$

where P is a product topology (by Definition (29.1)).

Proof. Define a map $\varphi : \prod_{i=1}^n X_i \to X_1 \times \cdots \times X_n$ by letting $\varphi(x) = (x_1, \ldots, x_n)$ for each $x \in \prod_{i=1}^n X_i$. Immediately, φ is a bijection.

To show that φ is continuous, let $G_1 \times \cdots \times G_n \in \tau$. Then, $\varphi^{-1}(G_1 \times \cdots \times G_n) = \bigcap_{i=1}^n p_i^{-1}(G_i) \in P$ and, therefore, φ is continuous by Theorem (25.1) 3.

If $\bigcap_{i=1}^{n} p_i^{-1}(V_i) \in P$, then $\varphi(\bigcap_{i=1}^{n} p_i^{-1}(V_i)) = V_1 \times \cdots \times V_n \in \tau$. Hence by Theorem (27.1) φ is a homeomorphism. $\square$

Henceforth we can always identify a product space $(\prod_{i=1}^{n} X_i, P)$ with the space $(X_1 \times \cdots \times X_n, \tau)$ of Theorem (29.1), which is notationally and conceptually simpler. Note that, in Theorem (29.1), the sets $G_1 \times \cdots G_n$ actually form a base for τ, since, for example,

$$(G_1 \times \cdots G_n) \cap (H_1 \times \cdots H_n) = (G_1 \cap H_1) \times \cdots \times (G_n \cap H_n).$$

Example 29.2. *Let $X_i = R$ for $i = 1, \ldots, n$. Then $(\prod_{i=1}^{n} X_i, P)$ is essentially the same as Euclidean $n-$space R^n.*

The projection maps $p_\beta : \prod_{\alpha \in \mathscr{A}} X_\alpha \to X_\beta$ are all continuous in the Tychonov product topology. Of course, the discrete topology on $\prod_{\alpha \in \mathscr{A}} X_\alpha$ also has this property, and is clearly the largest such topology. It turns out that the Tychonov topology is smallest in which each projection is continuous. A perhaps unexpected feature of the Tychonov topology is that each p_β is an open map as well.

Theorem 29.2.

1. *If $\beta \in \mathscr{A}$, then $p_\beta : \prod_{\alpha \in \mathscr{A}} X_\alpha \to X_\beta$ is a (P, T_β) continuous surjection.*
2. *If M is a topology on $\prod_{\alpha \in \mathscr{A}} X_\alpha$, and if p_β is (M, T_β) continuous for each $\beta \in \mathscr{A}$, then $P \subset M$.*
3. *p_β is an open map for each $\beta \in \mathscr{A}$.*

Proof.

1. Let $\beta \in \mathscr{A}$. Immediately, p_β is a surjection. If $G \in T_\beta$, then $p_\beta^{-1}(G) \in P$ by Definition (29.1), hence p_β is continuous.
2. Suppose $p_\beta : \prod_{\alpha \in \mathscr{A}} X_\alpha \to X_\beta$ is (M, T_β) continuous for each $\beta \in \mathscr{A}$. Then, $\{p_\beta^{-1}(V_\beta) \mid \beta \in \mathscr{A} \wedge (V_\beta \in T_\beta)\} \subset M$ by Definition (25.1). Then by Definition (29.1) and Theorem (24.1) $P \subset M$.

3. Let $\beta \in \mathscr{A}$. Note that, if $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ is a basic $P-$open set, then,

$$p_\beta(\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})) = \begin{cases} U_{\alpha_i} & \text{if } \beta = \alpha_i \text{ for some } i,\ 1 \leq i \leq n \\ X_\beta & \text{if } \beta \neq \alpha_i \text{ for each } i, \ldots, n \end{cases}$$

Thus, $p_\beta(b) \in T_\beta$ for each basic $P-$open set b. If now V is any $P-$open set, then there is some set C of basis open sets with $V = \cup C$. Then, $p_\beta(V) = p_\beta(\cup C) = \bigcup_{\beta \in C} p_\beta(b) \in T_\beta$.

□

Sometimes Theorem (29.2) 2 is used as the definition of the product topology. In this approach, P is by definition the intersection of all topologies on $\prod_{\alpha \in \mathscr{A}} X_\alpha$ in which each projection is continuous. one then proves that the sets $\{p_\beta^{-1}(V_\beta)$ constitute a base for the topology. This approach is motivated by a more general problem in topology: given maps $f_\alpha : X \to Y_\alpha$, and topologies M_α on Y_α, find the smallest topology T on X such that each f_α is continuous.

30 Slices in Cartesian Product Topology.

Given a point f in $\prod_{\alpha \in \mathscr{A}} X_\alpha$, and some $\beta \in \mathscr{A}$, the subset of $\prod_{\alpha \in \mathscr{A}} X_\alpha$ consisting of all g with $g(\alpha) = f(\alpha)$ whenever $\alpha \neq \beta$, may be visualized as a space parallel to the coordinate space X_β. For example, in $R^2 = R \times R$, the set of points $(x, 3)$ constitutes a space parallel to one copy of R i.e. a line parallel to the $x-$axis. It is not surprising that such a parallel space, or slice, is homeomorphic to X_β. This means that each X_β may be thought of as a subspace of $\prod_{\alpha \in \mathscr{A}} X_\alpha$, a fact which is extremely useful when we know something about the product space and wish to study the individual coordinate spaces.

Definition 30.1. Let $\prod_{\alpha \in \mathscr{A}} X_\alpha$ be an arbitrary cartesian product, and $x^o = \{x_\alpha^o\}$ a given point. For each index β, the set

$$S(x^o; \beta) = X_\beta \times \prod \{x_\alpha^o \mid \alpha \neq \beta\} \subset \prod_{\alpha \in \mathscr{A}} X_\alpha$$

is called the slice in $\prod_{\alpha \in \mathscr{A}} X_\alpha$ through x^o parallel to X_β

Example 30.1. *In $X^3 = X \times X \times X$, with $x^o = (x_1^o, x_2^o, x_3^o)$, the slice $S(x^o; 1) = \{(x_1^o, x_2^o, x_3^o) \mid x \in R\}$, and so is aline parallel to the $x-$axis going through x^o.*

Theorem 30.1.
The map $s_\beta : X_\beta \to S(x^o;\beta)$ given by

$$x_\beta \to x_\beta \times \prod \{x^o_\alpha \mid \alpha \neq \beta\}$$

is a homeomorphism of X_β with the subspace $S(x^o;\beta) = s$.

31 Connectedness.

Intuitively, a space is connected if it does not consist of two separate pieces. This simple idea has had important consequences in topology and has led to highly sophisticated algebraic techniques for distinguishing between spaces.

Definition 31.1. A topological space Y is connected if it is not the union of two nonempty disjoint open sets. A subset $B \subset Y$ is connected if it is connected as a subspace of Y.

Example 31.1. *Sierpinski space is connected: the only possible decomposition is 0,1, and 1 is not open. The discrete space 2 is not connected.*

Example 31.2. *The real number system with the upper-limit topology (i.e. topology generated by all sets of the form $\{x \mid x > a\}$ and $\{x \mid x \leq b\}$, therefore having the sets $]a, b]$ as basis) is not a connected space, since $\{x \mid x > a\}$ and $\{x \mid x \leq b\}$ are both open sets.*

Example 31.3. *The rationals $Q \subset R$ considering as a subset of real with Euclidean topology are not connected, since $\{x \mid x > \sqrt{2}\} \cap Q$, $\{x \mid x < \sqrt{2}\} \cap Q$ is decomposition as required.*

Theorem 31.1. *The only connected subsets of R with Euclidean topology having more than one point are R and the interval (open, closed, or half-open).*

Proof is left as a homework.

The definition (31.1) can be formulated in handier fashion.

Proposition 31.1. The following properties are equivalent:

1. Y is connected.
2. The only two subsets of Y are both open and closed are $\emptyset$ and Y.
3. No continuous $f : Y \to 2$ is surjective.

Proof.
- (1) $\Rightarrow$ (2). If $G \subset Y$ is both open and closed, and $G \neq \emptyset$, Y, then $Y = G \cup \mathbf{C}G$ shows that Y is not connected.

- (2) $\Rightarrow$ (3). If $f : Y \to 2$ were a continuous surjection, then $f^{-1}(0) \neq \emptyset, Y$, and because 0 is open and closed in 2, $f^{-1}(0)$ is open and closed in Y.
- (3) $\Rightarrow$ (1) If $Y = A \cup B$, A, B disjoint nonempty sets, then A, B are also closed, and the characteristic function $c_A = Y \to 2$ is a continuous surjection.

$\square$

Connectedness is clearly a topological invariant; even more,

Theorem 31.2. *The continuous image of a connected set is connected. That is, if X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. The map $f : X \to f(X)$ is continuous; if $f(X)$ were not connected it would be, by Proposition (31.1), a continuous surjection $g : f(X) \to 2$, and then $g \circ f : X \to 2$ would also be a continuous surjection, contradicting the connectedness of X. $\square$

Theorem 31.3. *Let Y be any space. The union of any family of connected subsets having at least one point in common is also connected.*

Proof. Let $C = \bigcup\limits_{\alpha} A_\alpha$, $y_0 \in \bigcap_\alpha A_\alpha$, and $f : C \to 2$ continuous. Since each A_α is connected, no $f|A_\alpha$ is surjective, and because $y_0 \in A_\alpha$ for each α, $f(y) = f(y_0)$ for all $y \in A_\alpha$ and all α. Thus f cannot be surjective. $\square$

Example 31.4. *In contrast, the intersection of even two connected sets need not be connected. Furthermore, if all the A_i, $i \in Z^+$ are connected, and $A_1 \subset A_2, \cdots$, still $C = \bigcap\limits_{\alpha} A_\alpha$, need not be connected: let $Y = I^2 - \{(x, 0) \mid \frac{1}{3} \leq x \leq \frac{2}{3}\}$ and $A_n = \{(x, y) \in Y \mid y \leq \frac{1}{n}\}$.*

Theorem 31.4. *Let $A \subset Y$ be connected. then any set B satisfying $A \subset B \subset \bar{A}$ is also connected. In particular, the closure of a connected set is connected.*

Proof. Let $f : B \to 2$ be continuous; since A is connected, $f|A$ is not surjective. Nothing that $B = \bar{A} \cap B = \bar{A_B}$, the continuity of f on B shows $f(B) = f(\bar{A_B}) \subset \bar{f(A)} = f(A)$, so that f cannot be surjective. $\square$

Example 31.5. *Since $Y = \{(x, y) \mid y = \sin \frac{1}{x}, 0 < x \leq 1\} \subset R^2$ with Euclidean topology is a continuous image of $]0, 1]$, it follows from theorems (31.2)and (31.4) that $\bar{Y} = Y \cup \{(0, y) \mid -1 \leq y \leq 1\}$ is connected. Observe that even with omission of any subset of $\{(0, y) \mid -1 \leq y \leq 1\}$, the resulting set is still connected.*

Theorem 31.5. *Let $\{Y_\alpha \mid \alpha \in \mathscr{A}\}$ be any family of spaces. $\prod\limits_{\alpha \in \mathscr{A}} Y_\alpha$ is connected if and only if each Y_α is connected.*

Proof is left as a question for the midterm exam.

32 Application to Real Valued Functions

We obtain a generalization of the "intermediate value theorem" of analysis.

Theorem 32.1. *Each continuous real-valued function on a connected space X takes on all values between any two assumes.*

Proof. Since, $f : X \to R$, is continuous, $f(X) \subset R$ is connected according to Theorem (31.2), so by Theorem (31.1), $f(X)$ is an interval. Thus, if $f(x) = a$, $f(x') = b$, we have $[a, b] \subset f(X)$, and therefore for each c such that $a \leq c \leq b$, there is an x'' with $f(x'') = c$. □

From theorems (31.1) and (31.5) follows that R^n, I^n and I^∞ are connected; even more,

Theorem 32.2. *Let $n > 1$, and $B \subset R^n$ be countable (R^n is with Euclidean topology). Then $R^n - B$ is connected.*

Proof. We can assume that $0 \notin B$, otherwise we move the origin. According to Theorem (31.3), it suffices to show that the origin and each $x \in R^n - B$ are contained in a connected set lying in $R^n - B$. Draw $\vec{0x}$ and let l be any line segment (say, of length 1) intersecting $\vec{0x}$ at exactly one point, distinct from 0 and x. For each $z \in l$, let $l_z = \vec{0x} \cup \vec{zx}$; each l_z is a connected set, and any two have only 0 and x in common. At least one l_z must lie in $R^n - B$: for $l_z \cap B \neq \emptyset$ for each $z \in l$, then since the points of intersection for differing z are necessarily distinct, we would find that B has a subset in 1-to-1 correspondence with the points of l and consequently B would not be countable. □

The usual technique for distinguishing between spaces stems from the observation: If $h : X \cong Y$, then by removing a set A of prescribed topological type from X, the spaces $X - A$, and $Y - h(A)$ are also homeomorphic, so that they must have the same topological invariants.

Theorem 32.3. *R^1 and R^n, $n > 1$, are not homeomorphic.*

Proof. Assume that $h : R^n \cong R^1$; removing one point $a \in R^n$, we must have $h : R^n - a \cong R^1 - h(a)$, by Theorem (27.3). However, this is impossible by Theorem (31.2), since, $R^n - a$ is connected whereas $R^1 - h(a)$ is not. □

The theorem than R^n is not homeomorphic to R^m for $n \neq m$ is much deeper, involving more delicate topological invariants (although the technique is the same). Conserving I^n $(n > 1)$ and I^1, a proof similar to Theorem (32.3)shows that they are not homeomorphic. thus, though there is a bijective map of the *set* I^1 onto the set I^n, there is no bicontinuous bijection and, as we shall see later not even continuous bijection.

Theorem 32.4. *In R each closed interval is homeomorphic to $[-1, +1]$, each open interval to $]-1, +1[$, and each half-open interval to $]-1, +1]$. Furthermore, no two of these intervals are homeomorphic.*

Proof. Given an interval with end points a, b, a suitable one of the maps $x \to \frac{b+a}{2} \pm \frac{b-a}{2}x$ exhibits a homeomorphism. To see that none of three standard intervals are homeomorphic, note that we can remove 2, 0, 1 (respectively) points without destroying the connectedness. □

33 Components

A disconnected space can be decomposed uniquely into connected "components"; the number of components provides a rough indication of how "disconnected" a space is.

Definition 33.1. Let Y be a space, and $y \in Y$. The component $C(y)$ of y in Y is the union of all connected subsets of Y containing y.

It is evident from Theorem (31.3) that $C(y)$ is connected.

Example 33.1. *Let $Q \subset R$ be subspace of rationals. The component of each $y \in Q$ is the point y itself. Thus, even though Y does not have the discrete topology, the components may reduce to points. Y is called* totally disconnected *if $C(y) = y$ for each $y \in Y$.*

Example 33.2. *Let $Y \subset R^2$ be subspace consisting of the segments joining the origin 0 to the points $\{(1, 1/n) \mid n \in Z^+\}$, together with the segment $]\frac{1}{2}, 1]$ on the $x-$axis. As in Example (31.5) Y is connected, but $Y - \{0\}$ is not: in $Y - \{0\}$ the component of each point is the ray containing it.*

Theorem 33.1.

1. *Each component $C(y)$ is a maximal connected set in Y: there is no connected subset of Y that properly contains $C(y)$.*

2. *The set of all distinct components in Y form a* partition *of Y.*

3. *Each $C(y)$ is closed in Y.*

Proof.

1. follows from the definition.

2. If $C(y) \cap C(y') \neq \emptyset$, then by Theorem (31.3), $C(y) \cap C(y')$ is connected, contradicting the maximality of $C(y)$.

3. Since $C(y)$ is connected, so also is $C\bar{(y)}$ by Theorem (31.4); by the maximality of $C(y)$ we must have $C\bar{(y)} \subset C(y)$, so that $C(y)$ is closed.

□

The number and structure of each component in a space Y is a topological invariant:

Theorem 33.2. *Let $f : X \to Y$ be continuous. Then the image of each component of X must lie in a component of Y. Furthermore, if $h : X \cong Y$, then h induces a 1-to-1 correspondence between the components of X and those of Y, corresponding ones being homeomorphic.*

Proof. If f is continuous, then $f(C(x)) \supset C(f(x))$ follows from Theorem (31.2), since $f(C(x))$ is a connected set in Y containing $f(x)$. If $h : X \cong Y$, then because h is bicontinuous and bijective, we have both $h(C(x)) \subset C(h(x))$ and $h^{-1}(C(h(x)))) \subset C(x)$, which shows that $h(C(x)) = C(h(x))$. The rest of the proof is trivial. □

34 Local Connectedness.

Definition 34.1. A space Y is locally connected if it has a basis consisting of connected (open) sets.

Example 34.1. *R^n is locally connected, since each ball $B(x; r)$ is connected. Furthermore, each interval in R is locally connected. For each $n \geq 0$, S^n is locally connected.*

Example 34.2. *A space may be locally connected, but not connected, as the discrete space 2 shows.*

Example 34.3. *A space may be connected, but not locally connected. Let Y be the space of Example (33.2). $y = (\frac{3}{4}, 0)$ and $U = B(y; \frac{1}{2}) \cap Y$. Then U, and any neighborhood $V(y) \subset U$, is not connected: For V must intersect a ray joining 0 to some $(1, 1/n)$ and it is trivial to verify that this intersection is both open and closed in V. Thus no basis for Y can consist only of connected sets.*

Theorem 34.1. *Y is locally connected if and only if the components of each* open *set are open sets.*

Proof. Let $G \subset Y$ be open, C a component of G, and $\{U\}$ a basis consisting of connected open sets. Given $y \in C$, then because $y \in G$, there is a U with $y \in U \subset C$; but since C is the component of y and U is connected, $y \in U \subset C$, showing that C is open (see Theorem (21.2)). For the converse, note that the family of all components of all open sets in Y is a basis as required. □

Example 34.4. *Observe that Theorem (34.1) need not be true for nonopen sets, as $\{0\} \cup \{1\} \subset R$ shows.*

Example 34.5. *Let Y be the space considered in Example (31.5), and $Z = Y \cup \{0\}$. Then Z is not locally connected, since the components of $Z \cap \{(x, y) \mid y < \frac{1}{2}\}$ are not open in Z.*

Proposition 34.1. A Cartesian product $\prod_{\alpha \in \mathscr{A}} Y_\alpha$ is locally connected if and only if all the Y_α are locally connected, and all but at most finitely many are also connected.

Example 34.6. *The hypothesis that all but most finitely many Y_α be connected is essential: If $A_n = \{0, 2\}$, we have seen $\prod_n A_n$ is totally disconnected. But though each A_n is locally connected, $\prod_n A_n$ is not: Its components are its points, and since $\prod_n A_n$ is not discrete, none is an open set.*

Example 34.7. *Local connectedness is evidently a topological invariant, and therefore it can be used in questions of nonhomeomorphism. Thus the space Z of Example (34.5) cannot be homeomorphic to any interval in R.*

Example 34.8. *Local connectedness is not invariant under the continuous maps. Let X be the discrete space $\{0, 1, 2, \cdots\}$, Y the subspace $0 \cup \{\frac{1}{n} \mid n = 1, 2, \cdots\}$ of R and $f : X \to Y$ the map $f(0) = 0,\ f(n) = \frac{1}{n}$. Then X is locally connected and f is a continuous bijection; but Y is not locally connected.*

35 Path-Connectedness.

For the most purposes of analysis, the natural notion of connectedness is joining by the path.

We can define a curve in space Y to be continuous image of the unit interval I. A *path* in Y is a continuous mapping $f : I \to Y$, rather than the image $f(I)$ in Y. Thus, a path is a continuous function, whereas a curve is a subset of Y; we shall see later a reason for this distinction between paths and curves. If $f : I \to Y$ is a path in Y, we call $f(0) \in Y$ the initial starting point, and the $f(1) \in Y$ the terminal (or end) point, of the path f, and say that f runs from $f(0)$ to $f(1)$, or joins $f(0)$ to $f(1)$. If f runs from $f(0)$ to $f(1)$, it is clear that the mapping $t- > f(1-t)$, $t \in I$, is a path in Y running from $f(1)$ to $f(0)$.

Definition 35.1. A space Y is path-connected (or: pairwise-connected) if each pair of its points can be joined by a path.

Example 35.1. *R^n and S^n $(n \geq 1)$ are path-connected. For any countable $B \subset R^n$, $R^n - B$ is also path-connected.*

Example 35.2. *Sierpinski space is path connected: the characteristic function of $1 \in I$, regarded as a map $I \to \mathscr{J}$, is a path joining 0 to 1.*

Example 35.3. *A discrete space having more than one point is never path-connected. Every indiscrete space is path-connected.*

A trivial but useful reformulation of Definition (35.1) is given in

Proposition 35.1. Let Y be a topological space, and $y_0) \in Y$ any element. Y is path-connected if and only if each $y \in Y$ can be joined to y_0 by a path.

Proof. If Y is path-connected, the condition if trivially true. Conversely, assume that the condition is satisfied and that $y, y' \in Y$ have been given. Let $f : I \to Y$ run from y to y_0 and $g : I \to Y$ from y_0 to y'; then

$$\phi(x) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is continuous (because at $t = \frac{1}{2}$, we have $f(1) = g(0) = y_0$) and is a path running from y to y'. □

The general relation of connectedness and path-connectedness is

Theorem 35.1. *Each path-connected space is connected. but a connected space need not be path-connected.*

Proof. Since the continuous image of I is connected, the assertion follows from Theorem (31.4) and Theorem (35.3). The following example shows that the converse is not generally true. □

Example 35.4. *Let Y be the space of Example (31.5); we have seen that $\bar{Y}$ is connected. However, $\bar{Y}$ is not path-connected: there is no path joining $(0,0)$ to the point $(\frac{1}{\pi}, 0)$. Proof of this statement is left as a homework.*

It is evident that path-connectedness is a topological invariant: Indeed, the continuous image of a path-connected space is path-connected. Furthermore, the union of any family of path-connected spaces having a point in common is, by Theorem (35.1), also path-connected. However, the *closure* of a path-connected set need not be path-connected.
Because of the property of unions, we can define path-connected components as maximal path-connected subsets; as before, the path components partition the space; indeed, from Theorem (35.1), the path components partition the components. However, the path components need not be closed subsets of the space.
To determine when path-connectedness and connectedness are equivalent, we need

Proposition 35.2. The following two properties of a space Y are equivalent:

1. Each path component is open (and therefore also closed).
2. Each point of Y has a path-connected neighborhood.

Proof.

- (1) $\Rightarrow$ (2) is clear, using the path component containing the given point.
- (2) $\Rightarrow$ (1) Lt K be any path component, and let $x \in K$. Since x has a path-connected neighborhood U, and since K is a maximal path-connected set containing x, $x \in U \subset K$, providing that K is open. Nothing that $\mathbf{C}K$ is the union of the remaining (open) path components, K is also closed.

□

Theorem 35.2. *Y is a path-connected if and only if it is connected, and each $y \in Y$ has a path-connected neighborhood.*

Proof. Since path-connectedness implies connectedness, and Y is a path-connected neighborhood of each point, only the converse requires proof. For this, we find from Proposition (35.2) 1) that each path component is both open and closed in Y; since Y is connected, this path component must therefore be Y. □

This yield an important

Corollary 35.1. *An open set in R^n (or S^n) is connected if and only if it is path-connected.*

Proof is left as a homework.