MA651 Topology. Lecture 6. Separation Axioms.

This text is based on the following books:

- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*
- *"Counterexamples in Topology" by Lynn A. Steen and J. Arthur Seebach, Jr.*
- *"Topology" by James Dugundgji*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

36 Separation Axioms.

So far, our only requirements for a topology has been that it satisfy the axioms. From now on, we will impose increasingly more severe additional conditions on it. With each new condition, we will determine the invariance properties of the resulting topology: by this we mean:

- Whether the topology is invariant under open or closed maps rather than only homeomorphisms.
- Whether the additional properties are inherited by each subspace topology.
- Whether the additional properties are transmitted to cartesian products.

In this lecture we will require of a topology that it "separate" varying types of subsets. The separation axioms T_i stipulate the degree to which distinct points or closed sets may be *separated by open sets*. These axioms are statements about the richness of topology. They answer questions like, "Are there enough open sets to tell points apart?" and "Are there enough open sets to tell points from closed sets?".

Definition 36.1. (T_i axioms). Let $(X, \mathscr{T})$ be a topological space.

T_0 axiom : If a, b are two distinct elements in X, there exists an open set $U \in \mathscr{T}$ such that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$ (i.e. U containing exactly one of these points).

T_1 axiom : If $a, b \in X$ and $a \neq b$, there exist open sets $U_a, U_b \in \mathscr{T}$ containing a, b respectively, such that $b \notin U_a$, and $a \notin U_b$.

T_2 axiom : If $a, b \in X$, $a \neq b$, there exist disjoint open sets $U_a, U_b \in \mathscr{T}$ containing a, b respectively.

T_3 axiom : If A is a closed set and b is a point in X such that $b \notin A$, there exist disjoint open sets $U_A, U_b \in \mathscr{T}$ containing A and b respectively.

T_4 axiom : If A and B are disjoint closed sets in X, there exist disjoint open sets $U_A, U_B \in \mathscr{T}$ containing A and B respectively.

T_5 axiom : If A and B are separated sets in X, there exist disjoint open sets $U_A, U_B \in \mathscr{T}$ containing A and B respectively.

If $(X, \mathscr{T})$ satisfies a T_i axiom, X is called a T_i space. A T_0 space is sometimes called a *Kolmogorov space* and a T_1 space, a Fréchet space. A T_2 is called a Hausdorff space.

Each of axioms in Definition (36.1) is independent of the axioms for a topological space; in fact there exist examples of topological spaces which fail to satisfy any T_i. But they are not independent of each other, for instance, axiom T_2 implies axiom T_1, and axiom T_1 implies T_0.

More importantly than the separation axioms themselves is the fact that they can be employed to define successively stronger properties. To this end, we note that if a space is both T_3 and T_0 it is T_2, while a space that is both T_4 and T_1 must be T_3. The former spaces are called regular, and the later normal.

Specifically a space X is said to be *regular* if and only if it is both a T_0 and a T_3 space; to be *normal* if and only if it is both a T_1 and T_4 space; to be completely normal if and only if it is both a T_1 and a T_5 space. Then we have the following implications:

$$\text{Completely normal} \Rightarrow \text{Normal} \Rightarrow \text{Regular} \Rightarrow \text{Hausdorff} \Rightarrow T_1 \Rightarrow T_0$$

The use of terms "regular" and "normal" is not uniform throughout the literature. While some authors use these terms interchangeably with "T_3 space" and "T_4 space" respectively, others (especially in Russian textbooks) refer to our T_3 space as a "regular" space and vice versa, and similarly permute "T_4 space" and "normal". This allows the successively stronger properties to correspond to increasing T_i axioms.

Example 36.1. *An antidiscrete space X containing more than one point does not satisfy to axiom T_0.*

Example 36.2. *Sierpinski space,* $X = \{0, 1\}$ *and* $\mathscr{T} = \{\emptyset, \{0\}, \{0, 1\}\}$*, satisfies the axiom* T_0*, but does not satisfy* T_1*: there does not exist an open set* U *containing the point 1 and not containing the point 0.*

Example 36.3. *This example is important for algebraic geometry. Let* A *be a commutative ring (i.e.* A *has two binary operations, such that addition makes* A *a commutative group (i.e. a set with a binary associative and commutative operation such that the operation admits an identity element and each element of the set has an inverse element for the operation.) and multiplication is associative and distributes over addition and the commutative law also holds for multiplication* $(a \times b = b \times a.)$ *with a unit and* X *the set of all prime ideals of* A *(i.e. such ideals which have the following two properties: whenever* a, b *are two elements of* A *such that their product* $a \times b$ *lies in the prime ideal* P*, then* a *is in* P *or* b *is in* P*, and at the same time* P *is not equal to the whole ring* A*). For any* $a \in A$*, let* X_a *denote the set of all prime ideals in* A *which do not contain* a*. It is clear that* $X_a \cap X_b = X_{ab}$ *for all* $a, b \in X$*,* $X_0 = \emptyset$ *and* $X_1 = X$*. Consequently, the collection* $\mathscr{B} = \{X_a \mid a \in A\}$ *is a base of a topology* $\mathscr{T}$ *on* A*. This topology is called the* spectral *or* Zariski topology*.*
The topological space $(X, \mathscr{T})$ *is called the* prime spectrum of the ring A *and is denoted Spect*(A)*. The closure of a one point set* $\{x\}$ *in Spec*(A) *consists of all prime ideals* $y \in X =$*Spec*(A) *containing* x*. It follows that the space* $(X, \mathscr{T})$ *satisfies the separation axiom* T_0*, but not* T_1*, since the only closed points in* X *are the maximal ideals of the ring* A*.*

Example 36.4. *Let* X *be an infinite set and let the topology* $\mathscr{T}$ *consist of the empty set and all subsets of* X *whose complements are finite. Any two nonempty open sets in this set intersect. At the same time all one point sets in* $(X, \mathscr{T})$ *are closed. Hence the space* $(X, \mathscr{T})$ *satisfies axiom* T_1 *but nit axiom* T_2*.*

37 Hausdorff Spaces.

Theorem 37.1. *Let* $(X, \mathscr{T})$ *be a topological space. Then the following statements are equivalent:*

1. *(*T_2 *axiom). Any two distinct points of* X *have disjoint neighborhoods.*
2. *The intersection of the closed neighborhoods of any point of* X *consists of that point alone.*
3. *The diagonal of the product space* $X \times X$ *is a closed set.*
4. *For every set* I*, the diagonal of the product space* $Y + X^I$ *is closed in* Y*.*
5. *No filter on* X *has more than one limit point.*
6. *If a filter* $\mathscr{F}$ *on* X *converges to* x*, then* x *is the only cluster point of* $\mathscr{F}$*.*

Proof. We will proof the implications:

$$(1) \Rightarrow (2) \Rightarrow (6) \Rightarrow (5) \Rightarrow (1)$$

$$(1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1)$$

$(1) \Rightarrow (2)$: If $x \neq y$ there is an open neighborhood U of x and an open neighborhood V of y such that $U \cap V = \emptyset$; hence $y \notin \bar{U}$.

$(2) \Rightarrow (6)$: Let $x \neq y$; then there is a closed neighborhood V of x such that $y \notin V$, and by hypothesis there exists $M \in \mathscr{F}$ such that $M \subset V$; thus $M \cap \mathbf{C}V = \emptyset$. But $\mathbf{C}V$ is a neighborhood of y; hence y is not a cluster point of $\mathscr{F}$.

$(6) \Rightarrow (5)$: Clear, since every limit point of a filter is also a cluster point.

$(5) \Rightarrow (1)$: Suppose $x \neq y$ and that every neighborhood V of x meets every neighborhood W of y. Then the sets $V \cap W$ form a basis of a filter which has both x and y as limits points, which is contrary to hypothesis.

$(1) \Rightarrow (4)$: Let $(x) = (x_i)$ be a point of X^I which does not belong to the diagonal $\triangle$. Then there are at least two indices λ, μ such that $x_\lambda \neq x_\mu$. Let V_λ (respectively V_μ) be a neighborhood of x_λ (respectively x_μ) in X, such that $V_\lambda \cap V_\mu = \emptyset$; then the set $W = V_\lambda \times V_\mu \times \prod_{i \neq \lambda, \mu} X_i$ (where $X_i = X$ if $i \neq \lambda, \mu$) is a neighborhood of x in X^I which does not meet $\triangle$. Hence $\triangle$ is closed in X^I.

$(4) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: If $x \neq y$ then $(x, y) \in X \times X$ is not in the diagonal $\triangle$, hence there is a neighborhood V of x and a neighborhood W of y in X such that $(V \times W) \cap \triangle = \emptyset$, which means that $V \cap W = \emptyset$.

□

Let $f : X \to Y$ be a mapping of a set X into a Hausdorff space Y; then it follows immediately from Theorem (37.1) that f has at most one limit with respect to a filter $\mathscr{F}$ on X, and that if f has y as a limit with respect to $\mathscr{F}$, then y is the only cluster point of f with respect to $\mathscr{F}$.

Proposition 37.1. let f, g be two continuous mappings of a topological space X into a Hausdorff space Y; then the set of all $x \in X$ such that $f(x) = g(x)$ is closed in X.

Proof is left as a homework.

Corollary 37.1. (Principle of extension of identities). *Let f, g be two continuous mappings of a topological space X into a Hausdorff space Y. If $f(x) = g(x)$ at all points of a dense subset of X, then $f = g$.*

In other words, a continuous map of X into Y (Hausdorff) is uniquely determined by its values at all points of a dense subset of X.

Corollary 37.2. *If f is a continuous mapping of a topological space X into a Hausdorff space Y, then the graph of f is closed in $X \times Y$*

For this graph is the set of all $(x, y) \in X \times Y$ such that $f(x) = y$, and the two mappings $(x, y) \to y$ and $(x, y) \to f(x)$ are continuous.

The invariance properties of Hausdorff topologies are:

Theorem 37.2.

1. *Hausdorff topologies are invariant under closed bijections.*
2. *Each subspace of a Hausdorff space is also a Hausdorff space.*
3. *The cartesian product $\prod\{X_\alpha \mid \alpha \in \mathscr{A}\}$ is Hausdorff if and only if each X_α is Hausdorff.*

Proof.

1. Since a closed bijection is also an open map, the images of disjoint neighborhoods are disjoint neighborhoods, and the result follows.
2. Let $A \subset X$ and $p, q \in A$; since there are disjoint neighborhoods $U(p)$, $U(q)$ in X, the neighborhoods $U(p) \cap A$ and $U(q) \cap A$ in A are also disjoint.
3. Assume that each X_α is Hausdorff and that $\{p_\alpha\} \neq \{q_\alpha\}$; then $p_\alpha \neq q_\alpha$ for some α, so choosing the disjoint neighborhoods $U(p_\alpha)$, $U(q_\alpha)$ gives the required disjoint neighborhoods $\langle U(p_\alpha)\rangle$, $\langle U(q_\alpha)\rangle$ in $\prod\limits_\alpha X_\alpha$. Conversely, if $\prod\limits_\alpha X_\alpha$ is Hausdorff, then each X_α is homeomorphic to some slice in $\prod\limits_\alpha X_\alpha$, so by (2) (since the Hausdorff property is a topological invariant), X_α is Hausdorff.

□

Proposition 37.2. If every point of a topological space X has a closed neighborhood which is a Hausdorff subspace of X, then X is Hausdorff.

Proof is left as a homework.

38 Regular Spaces.

The Hausdorff separation axiom (Theorem (37.1)) was introduced to insure uniqueness of limits (and, in fact, is equivalent to it). Certain problems, however, require stronger separation axioms if one to have any hope of success. The problem of existence of continuous extension is typical of this.

Suppose A is a subspace of X, and f is continuous on A to Y. Can f be extended continuously to some $g : X \to Y$? This is among the most difficult and interesting problems in topology.

In general, the answer is no, even when the spaces are Hausdorff (for example, $X = R$ with Euclidean topology, $A = \{x \mid x < 0\}$ and $f(x) = 1/x$ for $x \in A$. However, with a slightly stronger separation axiom on Y, reasonable conditions assuring a positive solution is possible. This result is Theorem (38.1), and may be thought as the motivation for considering the regularity.

Proposition 38.1. The following properties of a topological space X are equivalent:

1 The set of closed neighborhoods of any point of X is a fundamental system of neighborhoods of the point.

2 (T_3 axiom). Given any closed subset F of X and any point $x \notin F$ there is a neighborhood of x and a neighborhood of F which do not intersect.

Proof.

$1 \Rightarrow 2$: If F is closed and $x \notin F$, then there is a closed neighborhood V of x contained in the neighborhood $\mathbf{C}F$ of x; V and $\mathbf{C}V$ are neighborhoods of x and F respectively, and have no point in common.

$2 \Rightarrow 1$: If W is an open neighborhood of $x \in X$, then there is a neighborhood U of x and a neighborhood V of $\mathbf{C}W$ which are disjoint, and therefore $\bar{U} \subset W$.

□

Definition 38.1. A topological space is said to be regular if it is Hausdorff and satisfies axiom T_3; its topology is then said to be regular.

Remarks:

1 We can say than that the space is T_0 instead of Hausdorff in Definition (38.1).

2 A regular space is distinguished by its ability to separate closed sets from points. This is stronger than the T_2 separation axiom, and on the face seems to imply T_2 axiom by choosing $A = \{y\}$, where $y \neq x$. This reasoning is correct if singletons are always closed in X, but

this not need to be the case. However, in spaces where $\{y\}$ is closed for each point y, then of course only T_3 is needed in the definition of regularity. The reason of including T_2 (or T_0) in definition (38.1) are to insure the uniqueness of limits.

Example 38.1. *E^n, and, more generally, any metric space is regular.*

Example 38.2. *Discrete spaces are regular.*

Example 38.3. *A Hausdorff space need not be regular. Let R be the set of real numbers, $\mathscr{T}$ the usual Euclidean topology on R, Q the set of rationals, and $\mathscr{T}'$ the topology on R generated by Q and the sets in $\mathscr{T}$. Then $\mathscr{T} \subset \mathscr{T}'$, so $(R, \mathscr{T}')$ is a Hausdorff space. But $(R, \mathscr{T})$ is not regular.*

Note that the $\mathscr{T}'$-open sets are $Q \cap$ (Q intersect $\mathscr{T}$-open sets) and ($\mathscr{T}$-open sets) $\cup$ (Q intersect $\mathscr{T}$-open sets). Choose any rational number, say 0. Then, $0 \notin R-Q$. If $(R, \mathscr{T}')$ were regular, there would be disjoint $\mathscr{T}'$-open U and V with $R-Q \subset U$ and $0 \subset V$. Now, for some $\mathscr{T}$-open A and B, $U = A \cup (Q \cap B)$. And since each $\mathscr{T}$-open set intersects $R-Q$, then for some $\mathscr{T}$-open C, $V = Q \cap C$.

Let $\varepsilon > 0$ such that $Q \cap]-\varepsilon, \varepsilon[\subset Q \cap C$. Choose an irrational number $\xi \in]-\varepsilon, \varepsilon[$. Then, $\xi \in R - Q \subset A \in \mathscr{T}$. Since A is $\mathscr{T}$-open, there is some $\delta > 0$ with $]\xi - \delta, \xi + \delta[\subset]-\varepsilon, \varepsilon[\cap A$. Choose a rational number r in $]\varepsilon - \delta, \varepsilon + \delta[$. Then, $r \in A \subset U$, and also $r \in Q \cap]-\varepsilon, \varepsilon[\subset V$. Then, $U \cap V \neq \emptyset$, a contradiction.

Proposition 38.2. Every subspace of a regular space is regular.

Proof. Let A be a subspace of a regular space X. Since X is Hausdorff so is A; on the other hand, every neighborhood of a point $x \in A$ with respect to A is of the form $V \cap A$, where V is a neighborhood of x in X. Since X is regular there is a neighborhood W of x in X which is closed in X and contained in V; $W \cap A$ is then a neighborhood of x in A, closed in A and contained in $V \cap A$. Hence the result. □

The converse statement is also true:

Proposition 38.3. If every point x of a topological space X has a closed neighborhood which is a regular subspace of X, then X is regular.

Proof is left as a homework.

Theorem 38.1. *Let X be a topological space, A a dense subset of X, $f : A \to Y$ a mapping of A into a regular space Y. A necessary and sufficient condition for f to extend to a continuous mapping $\bar{f} : X \to Y$ is that. for each $x \in X$, $f(y)$ tends to a limit in Y when y tends to x while remaining in A. The continuous extension $\bar{f}$ of f to X is then unique.*

Proof. The uniqueness of $\bar{f}$ follows from the principle of extension of identities (37.1). It is clear that the condition is necessary, for if $\bar{f}$ is continuous on X, then for each $x \in X$ we have

$$\bar{f}(x) = \lim_{y \to x,\, y \in A} \bar{f}(y) = \lim_{y \to x,\, y \in A} f(y)$$

Conversely, suppose that the condition is satisfied and *define*

$$\bar{f} = \lim_{y \to x,\, y \in A} f(y)$$

for each $x \in X$; $\bar{f}(x)$ is a well-defined element of Y, since Y is Hausdorff. We have to show that $\bar{f}$ is *continuous* at each point $x \in X$. let then V' be a *closed* neighborhood of $\bar{f}(x)$ in Y; then by hypothesis there is an *open* neighborhood V of x in X such that $f(V \cap A) \subset V'$. Since V is a neighborhood of each of its points, we have

$$\bar{f}(z) = \lim_{y \to x,\, y \in V \cap A} f(y)$$

for each $z \in V$, and from this it follows that $\bar{f}(z) \in \overline{f(V \cap A)} \subset V'$, since V' is closed. The result now follows from the fact that the closed neighborhoods of $f(x)$ form a fundamental system of neighborhoods of $f(x)$ in Y.

□

The mapping $\bar{f}$ is said to be obtained by *extending f by continuity to X.*

It important to notice that the statement of Theorem (38.1) the hypothesis that Y is regular cannot be weakened without imposing additional restrictions on X, A or f.

39 Normal Spaces.

We have seen how certain problems give rice to separation axioms of varying strength. The T_3 axiom enables us to prove a reasonable theorem of continuous extensions. In similar fashion, normal spaces were devised to treat questions concerning continuous functions which were found to lie beyond the scope of regular spaces. This type of separation is stronger than regularity and is given by

Definition 39.1. A Hausdorff space is normal if each pair of disjoint closed sets have disjoint open neighborhoods; its topology is then said to be normal.

Example 39.1. *Any metrizable space X is normal (for instance E^n). Let ρ be a metric generating the topology on X, and let A and B be disjoint, nonempty closed subsets of X. For $x \in A$ and $y \in B$ we set $V_x = O_{\varepsilon_x}(x) = \{z \in X : \rho(x,z) < \varepsilon_x\}$ where $\varepsilon_x = \rho(x,B)/3 > 0$ and $U_y = O_{\varepsilon_y}(y) = \{z \in X : \rho(y,z) < \varepsilon_y\}$, where $\varepsilon_y = \rho(y,A)/3 > 0$. The sets $V = \bigcup\{V_x : x \in A\}$ and $U = \bigcup\{U_y : y \in B\}$ are disjoint neighborhoods of the sets A and B respectively. This follows from the triangle axiom.*

Definition (39.1) has several equivalent formulations:

Proposition 39.1.

1. X is normal.
2. For each closed A and open $U \supset A$ there is an open V with $A \subset V \subset \bar{V} \subset U$.
3. For each pair of disjoint closed sets A, B, there is an open U with $A \subset U$ and $\bar{U} \cap B = \emptyset$.
4. Each pair of disjoint closed sets have neighborhoods whose closures do not intersect.

Proof is left as a homework.

In the invariance properties, we meet a situation different from those met before. For instance, a subspace of a normal space can fail to be normal; that is axiom T_4 is not inherited be subspaces. This is one of the main inconveniences in dealing with the class of normal spaces. The following theorem summarize invariance properties of normal spaces:

Theorem 39.1.

1. *Normality is invariant under continuous closed surjections.*
2. *A subspace of a normal space need* not *be normal. However, a* closed *subspace is normal.*
3. *The cartesian product of normal spaces need* not *be normal. However, if the product is normal, each factor must be normal.*

Proof.

1. Let Y be normal and $p : Y \to Z$ be closed and continuous. Given disjoint closed A, B in Z, the normality of Y gives disjoint open sets with $p^{-1}(A) \subset U$, $p^{-1}(B) \subset V$. Because p is closed, Theorem (26.1) assures that there exist open $U_A \supset A$, $V_B \supset B$ such that $p^{-1}(U_A) \subset U$, $p^{-1}(V_B) \subset V$ and U_A, V_B are evidently the required disjoint neighborhoods of A and B.
2. An example of a non-normal subspace of a normal space will be considered later (in a lecture about compactness). The second assertion is immediate from the observation that a set closed in a closed subspace is also closed in the entire space.
3. The example of the first assertion (i.e. the cartesian product of normal spaces need not be normal) will be given later. The second statement follows from (2) and (1) (similarly to Theorem (37.2) (3)).

□

40 Urysohn's characterization of normality.

We now turn to the reason for considering the T_4 separation axiom.

Normal spaces came to the attention of Urysohn in the 1920's in connection with the following question: given the space $(X, \mathscr{T})$, are there "enough" real-valued continuous functions on X?. The word "enough" is purposely vague, but at the very last we would like a guarantee of the existence of some non-constant continuous function from X to E^1. The T_3 axiom does not provide such a guarantee. Urysohn was able to show that the T_4 axiom does.

More specifically, with each pair of disjoint, closed subsets of any normal space, Urysohn was able to associate a continuous function $f : X \to [0, 1]$ which separates A and B in the sense that f maps A to 0 and B to 1. It also turns out that, conversely, the existence of an Urysohn function for each pair of disjoint closed sets insures normality of the space.

The proof of Urysohn's Lemma is fairly deep, as might be expected of one fundamental results of set topology. We precede it by Lemma (40.1) which handles some of the more technical details. In proving Lemma (40.1) we make use of the easily established fact that, if $0 \leqslant x < y \leqslant 1$, then dyadic numbers (i.e., of the form $m/2^m$) d and d' can be found such that $x \leqslant d < d' \leqslant y$.

Lemma 40.1. *Let $(X, \mathscr{T})$ be a normal topological space. Let A and B be disjoint $\mathscr{T}$-closed sets. Then, there is a set $\{U_t \mid 0 \leqslant t \leqslant 1\}$ of $\mathscr{T}$-open sets such that:*

1. *$A \subset U_0$.*
2. *$U_1 \cap B = \emptyset$.*
3. *If $0 \leqslant x < y \leqslant 1$, then $\bar{U}_x \subset U_y$.*

Proof. Define $U_1 = X - B$. Then, $A \subset U_1$, and U_1 is $\mathscr{T}$-open and $U_1 \cap B = \emptyset$. By Proposition (39.1), there is some $\mathscr{T}$-open set U_0 such that $A \subset U_0 \subset \bar{U}_0 \subset U_1$.

We now proceed to fill in U_t when $0 < t < 1$. We first work with t dyadic, proceeding by induction.

Suppose n is a non-negative integer, and that the sets $U_{k/2^n}$ have been defined for integers k, $0 \leqslant k \leqslant 2^n - 1$ (note that U_0 and U_1 have already been defined). Thus, $\bar{U}_{j/2^n} \subset U_{i/2^n}$ for $0 \leqslant j < i \leqslant 2^n - 1$. We must define the sets $U_{i/2^{n+1}}$, $0 \leqslant i \leqslant 2^{n+1} - 1$. Note first that we need only consider odd values of i. If i is even. say $i = 2j$, then $0 \leqslant j \leqslant 2^n - 1$ and $i/2^{n+1} = j/2^n$, and $U_{j/2^n}$ has already been defined by the inductive hypothesis.

Suppose then that i is odd, say $i = 2k+1 \leqslant 2^{n+1}-1$, for some k, $0 \leqslant k \leqslant 2^n - 1$. Then, again, by Proposition (39.1), since $\bar{U}_{k/2^n} \subset U_{(k+a)/2^n}$, there is some $\mathscr{T}$-open set V such that $\bar{U}_{k/2^n} \subset V \subset \bar{V} \subset U_{(k+1)/2^n}$. Let $U_{(2k+1)/2^{n+1}} = V$. This defines $U_{i/2^{n+1}}$ for all i, $0 \leqslant i \leqslant 2^{n+1}-1$.

By induction, the sets U_d are now defined for all dyadic d.

Now let t be any number in $[0,1]$.

Define $U_t = \bigcup_{d \leqslant t} U_d$ the union being over all dyadics d in $[0,1]$ with $d \leqslant t$. If t is dyadic, then this definition agrees with that arrived at by induction.

Finally, suppose that $0 \leqslant x < y \leqslant 1$. Then there are dyadic numbers $i/2^n$ and $j/2^m$ with $x \leqslant j/2^m < i/2^n \leqslant y$. Now, $U_{i/2^n} \subset U_y$ by definition of U_y. Further, for all dyadic d with $0 \leqslant d \leqslant x$, we have $d \leqslant j/2^m$, so $U_d \subset U_{j/2^m}$, hence $U_x \subset U_{j/2^m}$. Then

$$\bar{U}_x \supset \bar{U}_{j/2^m} \supset U_{j/2^n} \subset U_y,$$

and the Lemma is proved. □

We now prove the main theorem, which traditionally is known as Urysohn's Lemma. It is understood in the proof that $[0,1]$ is considered as a subspace of Euclidean space E^1.

Theorem 40.1. *(Urysohn's Lemma). Let $(X, \mathscr{T})$ be a Hausdorff topological space. Then the following are equivalent:*

1. *$(X, \mathscr{T})$ is normal.*
2. *If A and B are disjoint non-empty $\mathscr{T}$-closed sets, then there exists a continuous $f : X \to E^1$, called a Urysohn function for A, B, such that:*
   - *(a) $0 \leqslant f(x) \leqslant 1$ for all $x \in X$*
   - *(b) $f(a) = 0$ for all $a \in A$*
   - *(c) $f(b) = 1$ for all $b \in B$*

Proof.

(2) $\Rightarrow$ (1) : If A and B are non-empty, disjoint, $\mathscr{T}$-closed subsets of X, then produce by (2) an Urysohn function f and note that $f^{-1}([0, \frac{1}{2}[)$ and $f^{-1}(]\frac{1}{2}, 1])$ are disjoint, $\mathscr{T}$-neighborhoods of A and B respectively.

(1) $\Rightarrow$ (2) : Assume (1). Let A and B be non-empty, disjoint, $\mathscr{T}$-closed sets. Let the $\mathscr{T}$-open sets U_t, $0 \leqslant t \leqslant 1$, be as given by Lemma (40.1). If $x \in X$, define:

$$f(x) = \begin{cases} 1 & \text{if } x \notin U_1 \\ inf\{t \mid x \in U_t\} & \text{if } x \in U_1 \end{cases}$$

Immediately, $f : X \longrightarrow [0,1]$. If $x \in B$, then $x \notin U_1$, so $f(x) = 1$. If $x \in A$, then $x \in U_0$ and $x \in U_1$, so $f(x) = 0$.

There remains to show that f is continuous.

Let $x \in X$ and let $\varepsilon > 0$. Consider three cases:

i) $f(x) = 0$
We may assume without loss of generality that $\varepsilon < 1$.
Then, $|f(y) - f(x)| = |f(y)| = f(y) \leqslant \varepsilon/2 < \varepsilon$ for $y \in U_{\varepsilon/2}$.

ii) $f(x) = 1$
Again, we may assume that $\varepsilon < 1$. If $y \in X - \bar{U}_{1-\varepsilon/2}$, then $y \notin U_j$ for $j \leqslant 1 - \varepsilon/2$. Then, $f(y) \geqslant 1 - \varepsilon/2$, so

$$1 - f(y) = |f(x) - f(y)| \leqslant \varepsilon/2 < \varepsilon$$

iii) $0 < f(x) < 1$
We may assume that $\varepsilon \leqslant f(x)$ and $f(x) + \varepsilon \leqslant 1$.
Let $y \in U_{f(x)+\varepsilon/2} \cap (X - \bar{U}_{f(x)-\varepsilon/2})$.
Since $y \in U_{f(x)+\varepsilon/2}$, then $y \notin B$, and $f(y) \leqslant f(x)+\varepsilon/2 < f(x)+\varepsilon$, so $f(y) - f(x) < \varepsilon$.
But also $y \in X - \bar{U}_{f(x)-\varepsilon/2}$, so $y \notin \bar{U}_j$ for $0 \leqslant j \leqslant f(x) - \varepsilon/2$.
Then, $(y) \geqslant f(x) - \varepsilon/2 > f(x) - \varepsilon$, so $f(y) - f(x) > -\varepsilon$.
Then, $-\varepsilon < f(y) - f(x) < \varepsilon$, so $|f(y) - f(x)| < \varepsilon$.

By cases *i*) through *iii*), f is continuous at $x \in X$, hence f is continuous.

□

Remarks:

1. The pair 0 and 1 can be obviously replaced by any (not necessary nonnegative) pair $\alpha < \beta$, since the continuity of f implies that of $\alpha + (\beta - \alpha)f$.
2. The Urysohn function f in Theorem (40.1) evidently satisfies $A \subset f^{-1}(0)$. The theorem does *not* assert that $A = f^{-1}(0)$; in fact, this is possible only for certain types of closed sets. This problem we will consider later.

41 Tietze's characterization of normality.

The very strong connection between the T_4 separation axiom and real continuous functions on X, suggested by Urysohn's Lemma, is very clearly revealed by the next theorem, which should be compared with Theorem (38.1). Theorem (41.1) says that normality is sufficient for the existence of a continuous extension of any real-valued function continuous on any closed subspace.

The converse is also true, so that normality is just the right condition for treating the extension problem on closed subspaces.

Theorem (41.1) is often known as Tietze's Extension Theorem, although there is some confusion about assigning credit here. Historically, Tietze seems to have been the first to define normality, in 1923. Urysohn's Lemma clarified its role in the theory of continuous functions. Tietze proved his extension theorem for metric spaces, which of course are normal, but it was Urysohn who generalized the result to arbitrary normal spaces. Thus Bourbaki refers to Theorem (41.1) as a theorem of Urysohn, and Dugundji as Tietze's Theorem. Alexandroff calls it Brouwer-Urysohn Theorem, and Stone the Lebesgue-Urysohn Theorem. It would appear to be fair to credit Tietze with the first proof for a wide class of spaces (metric spaces), and Urysohn with the proof of the general statement.

In the course of the proof, we shall use the Cauchy criterion for the convergence of real sequences, and also the fact that $\lim_{k\to\infty} |S_{n+k}-S_n| = |L-S_n|$ for any real sequence S if $\lim_{n\to\infty} S_n = L$. Preliminary Lemma (41.1) will absorb some if the technical details. In Lemma (41.1) and Theorem (41.1), all intervals $[a,b]$ are considered as subspaces of E^1.

Lemma 41.1. *Let $(X,\mathscr{T})$ be a normal space. Let F be a $\mathscr{T}$-closed subset of X and let $f : F \to [-1,1]$ be continuous. Then, there is a continuous $g : X \to [-\frac{1}{3},\frac{1}{3}]$ such that $|g(x)-f(x)| \leqslant \frac{2}{3}$ for each $x \in F$.*

Proof. Let $A = f^{-1}([-1,-\frac{1}{3}])$ and $B = f^{-1}([\frac{1}{3},1])$. Since f is continuous, A and B are $\mathscr{T}$-closed. Since $A \cap B = \emptyset$ and $\mathscr{T}$ is normal, there is by Urysohn's Lemma some continuous $h : X \to [0,1]$ such that $h(A) = \{0\}$ and $h(B) = \{1\}$.
Let $t(x) = \frac{2}{3}x - \frac{1}{3}$, for $0 \leqslant x \leqslant 1$. This defines an homeomorphism $t : [0,1] \to [-\frac{1}{3},\frac{1}{3}]$.
Let $g = t \circ h$. Then, $g : X \to [-\frac{1}{3},\frac{1}{3}]$ is continuous.
If $x \in A$, then $g(x) = t(h(x)) = t(0) = -\frac{1}{3}$. Since $f(x) \in [-1,-\frac{1}{3}]$, then $|f(x)-g(x)| \leqslant \frac{2}{3}$.
If $x \in B$, then $g(x) = t(1) = \frac{1}{3}$. Since $f(x) \in [\frac{1}{3},1]$, then $|f(x)-g(x)| \leqslant \frac{2}{3}$.
Finally, if $x \in F-(A\cup B)$, then $-\frac{1}{3} < f(x) < \frac{1}{3}$, so $|g(x)-f(x)| \leqslant \frac{2}{3}$. □

Theorem 41.1. *(Tietze-Urysohn) Let $(X,\mathscr{T})$ be a Hausdorff space. Then the following are equivalent:*

1. *$\mathscr{T}$ is normal.*

2. *If A is $\mathscr{T}$-closed and $f : A \to E^1$ is continuous, then, there is a continuous $F\colon X \to E^1$ such that $F|A = f$.*

Proof.

(2) $\Rightarrow$ (1) : Assume (2). Suppose that A and B are disjoint $\mathscr{T}$-closed sets. Define $f : A \cup B \to E^1$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Then f is easily seen to be continuous. By (2) produce a continuous extension F of f to X.

Then, $F^{-1}(]-\frac{1}{2}, \frac{1}{3}[)$ and $F^{-1}(]\frac{1}{2}, 2[)$ are disjoint $\mathscr{T}$-open neighborhoods of A and B respectively. Hence $\mathscr{T}$ is normal.

(2) $\Rightarrow$ (1) : Assume (1). Let A be $\mathscr{T}$-closed. Suppose $f : A \to E^1$ is continuous. We consider two cases and proceed in steps.

Case 1. f is bounded.

Then for some $M > 0$, $|f(x)| \leqslant M$ for each $x \in A$. We may assume without loss of generality that $f : A \to [-1, 1]$. For, $|(1/M)f(x)| \leqslant 1$ for each $x \in A$, and it is immediate that MF is a continuous extension of f if F is a continuous extension of $(1/M)f$.

Thus, suppose that $f : A \to [-1, 1]$.

i) If n is a non-negative integer, then there is a continuous $F_n : X \to [-1 + (\frac{2}{3})^{n+1}, 1 - (\frac{2}{3})^{n+1}]$ such that $|F_n(x) - f(x)| \leqslant (\frac{2}{3})^{n+1}$ for each $x \in A$ and, if K is a positive integer, $|F_m(x) - F_n(x)| \leqslant 2(\frac{2}{3})^{K+1}$ for $m, n \geqslant K$.
We produce the functions F_n by an inductive construction. By Lemma (41.1), there is a continuous $F_0 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $|F_0 - f(x)| \leqslant \frac{2}{3}$ for each $x \in A$.

Now suppose n is a non-negative integer, and F_n has been defined. Let $\varphi(x) = (\frac{3}{2})^{n+1}(f(x) - F_n(x))$ for each $x \in A$. Then, φ is continuous: $A \to [-1, 1]$. By Lemma (41.1), there is a continuous $\beta : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $|\beta(x) - \varphi(x)| \leqslant \frac{2}{3}$ for each $x \in A$.

Let $F_{n+1}(x) = F_n(x) + (\frac{2}{3})^{n+1}\beta(x)$ for each $x \in X$. F_{n+1} is continuous: $X \to [-1+(\frac{2}{3})^{n+2}, 1-(\frac{2}{3})^{n+2}]$.

Further, if $x \in A$, then

$$\begin{aligned}
|F_{n+1}(x) - f(x)| &= |F_n(x) + \left(\frac{2}{3}\right)^{n+1} \beta(x) - f(x)| \\
&= \left(\frac{2}{3}\right)^{n+1} |\beta(x) - \left(\frac{3}{2}\right)^{n+1} (f(x) - F_n(x))\,| \\
&= \left(\frac{2}{3}\right)^{n+1} |\beta(x) - \varphi(x)| \\
&\leqslant \left(\frac{2}{3}\right)^{n+1} \left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{n+2}
\end{aligned}$$

Finally, let $x \in X$ and let n be a positive integer. By the definition of F_{n+1}, $|F_{n+1}(x) - F_n(x)| = \left(\frac{2}{3}\right)^{n+1} |\beta(x)| \leqslant \left(\frac{2}{3}\right)^{n+1} \frac{1}{3} = 2^{n+1}/3^{n+2}$. By inducting we have, for $r \geqslant 1$,

$$\begin{aligned}
|F_{n+r}(x) - F_n(x)| &\leqslant \sum_{j=0}^{r-1} |F_{n+j+1}(x) - F_{n+j}(x)| \\
&\leqslant \sum_{j=0}^{r-1} \left(\frac{2^{n+j+1}}{3^{n+j+2}}\right) = \frac{2^{n+1}}{3^{n+2}} \sum_{j=0}^{r-1} \left(\frac{2}{3}\right)^j \\
&< \frac{2^{n+1}}{3^{n+2}} \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j = \left(\frac{2}{3}\right)^{n+1}
\end{aligned}$$

Then, for m, $n \geqslant K$,

$$\begin{aligned}
|F_m(x) - F_n(x)| &\leqslant |F_m(x) - F_K(x)| + |F_n(x) - F_K(x)| \\
&\leqslant \left(\frac{2}{3}\right)^{K+1} + \left(\frac{2}{3}\right)^{K+1} = 2\left(\frac{2}{3}\right)^{K+1}
\end{aligned}$$

ii) If $x \in X$ then $\lim_{n\to\infty} F_n(x)$ exists, and $-1 \leqslant \lim_{n\to\infty} F_n(x) \leqslant 1$.

Existence follows from i). Since $\left(\frac{2}{3}\right)^{K+1} \to 0$ as $K \to \infty$, then $\{F_n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence to E^1, hence converges. Since $-1 \leqslant F_n(x) \leqslant 1$ for each n, then $-1 \leqslant \lim_{n\to\infty} F_n(x) \leqslant 1$ also.

iii) Let $F(x) = \lim_{n\to\infty} F_n(x)$ for each $x \in X$.
Then, F is continuous on X to $[-1, 1]$.

All that requires proof is the continuity. Let $x \in X$ and $\varepsilon > 0$. Note that $|F(z) - F_n| = \lim_{K \to \infty} |F(n+K) - F_n| \leqslant 2\left(\frac{2}{3}\right)^{n+1}$ for each $n \in Z^+$ and $z \in X$.

Now, if $y \in X$, then for any $n \in Z^+$ we have

$$|F(x) - F(y)| \leqslant |F(x) - F_n(x)| + |F_n(x) - F_n(y)| + |F_n(y) - F(y)|$$

Choose n sufficiently large that $2\left(\frac{2}{3}\right)^{n+1} > \varepsilon/3$. Since F_n is continuous, there is a $\mathscr{T}$-neighborhood. V of x such that $|F_n(x) - F_n(y)| < \varepsilon/3$ if $y \in V$. Then, for each $y \in V$, $|F(x) - F(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$, implying that F is continuous at x, hence continuous.

iv) $F|A = f$.
Let $x \in A$ and $\varepsilon > 0$. Then, for each $n \in Z^+$,

$$|F(x) - f(x)| \leqslant |F_n(x) - F(x)| + |F_n(x) - f(x)| \leqslant |F_n(x) - F(x)| + \left(\frac{2}{3}\right)^{n+1}.$$

Choose n sufficiently large that $\left(\frac{2}{3}\right)^{n+1} < \varepsilon/2$ and $|F_n(x) - F(x)| < \varepsilon/2$.

Then, $|F(x) - f(x)| < \varepsilon$, implying $F(x) = f(x)$.

This completes the proof of the theorem in case 1.

Case 2. f is not bounded.

Define $g(x) = \frac{f(x)}{1+|f(x)|}$ for each $x \in A$
Please try to complete the proof by filling the details of the remaining steps.

v) g is continuous: $A \to [-1, 1]$.

vi) There exists a continuous $G : X \to [-1, 1]$ such that $G|A = g$ (apply case 1 to g).

vii) There is a continuous $h : X \to [0, 1]$ such that $h(a) = 1$ whenever $a \in A$ and $h(b) = 0$ whenever $G(b) = 1$ or $G(b) = -1$. (apply Urysohn's Lemma to A and $G^{-1}(\{-1\} \cup \{1\})$.)

viii) Let $F(x) = \frac{G(x)h(x)}{1-|G(x)h(x)|}$ Then F is continuous: $X \to E^1$ and $F|A = f$. This completes the proof of the theorem.

□