MA651 Topology. Lecture 7. Real numbers.

This text is based on the book "Linear Algebra and Analysis" by Marc Zamansky

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

42 Algebraic Laws

Definition 42.1. An internal composition law on a set E is a mapping of $E \times E$ into E.

Definition 42.2. An internal composition law on a set E is called associative if for all x, y, z in E we have:

$$(xTy)Tz = xT(yTz)$$

Definition 42.3. An internal composition law on E is called commutative if for every x and y in E we have

$$xTy = yTx$$

Definition 42.4. An element $a \in E$ is said to be regular for the internal law T if for all $x, y \in E$ the relations $aTx = aTy$ and $xTa = yTa$ imply $x = y$.

Definition 42.5. An element $e \in E$ is said to be a unit element for an internal law T on E if for all $x \in E$ we have

$$eTx = xTe = x$$

Theorem 42.1. *If e is a unit for an internal law T, it is unique.*

Proof is left as a homework.

Definition 42.6. Let T be an internal law on E *having a unit* e. An element $x \in E$ is said to possess an inverse for this law if there exists $x' \in E$ such that $xTx' = x'Tx = e$.

Definition 42.7. A set G is called a group if it has an internal law T having the three following properties:

(A) it is associative: $(xTy)Tz = xT(yTz)$

(N) it has a unit e: $eTx = xTe = x$

(S) every element x of G has an inverse $x' : xTx' = x'Tx = e$

A composition law with these properties is called a group law. If, further, the law T is commutative ($xTy = yTx$), the group is called commutative or Abelian.

Definition 42.8. A ring is a set A endowed with two internal composition laws, the first being that of an Abelian group, the second being associative, and distributive with respect to the first. If we write the first law additively and the second multiplicatively, then:
First Law:

(A) $(x + y) + z = x + (y + z)$

(N) $x + e = e + x = x$

(S) $x + (-x) = e$

(C) $x + y = y + x$

Second Law:
(A) $(xy)z = x(yz)$
Distributive Law:
(D) $(x + y)z = xz + yz$ and $z(x + y) = zx + zy$
If the second law is also commutative ($xy = yx$) A is called a *commutative ring.*
If the second law has a unit element ε ($x\varepsilon = \varepsilon x = x$), it is called a unit of A and A is called a *ring with unit.*

Definition 42.9. Let K be a ring and e the unit for the first law (the Abelian group law); let K^* be the set of elements of K other than e. If the second law on K is a group law on K^*, K is called a field.
First Law:

(A) $(x + y) + z = x + (y + z)$

(N) $x + e = e + x = x$

(S) $x + (-x) = e$

(C) $x + y = y + x$

Second Law:

(A) $(xy)z = x(yz)\ \forall x, y, z \in K$

(N) $x\varepsilon = \varepsilon x = x\ \forall x \in K^*$

(S) $xx^{-1} = x^{-1}x = \varepsilon\ \forall x \in K^*$

Distributive Law:
(D) $(x + y)z = xz + yz$ and $z(x + y) = zx + zy\ \forall x, y, z \in K$
If the second law is also commutative ($xy = yx$) K is called a *commutative field.*

Definition 42.10. An ordered group is a set G endowed with an Abelian group structure and an order structure related by the following condition:

$$\text{for all } z \in G,\ x \leqslant y \text{ implies that } x + z \leqslant y + z$$

Elements x such that $0 \leqslant x$ are called positive elements.

The property $x + z \leqslant y + z$ can be expressed: order is invariant under translations.

Definition 42.11.
1. A Riesz group is an ordered group G such that for all $x, y \in G$, $\sup(x, y)$ and $\inf(x, y) \in G$.
2. The positive (negative) part of $x \in G$, written x^+ (x^-) is the element $\sup(x, 0)$ ($\sup(-x, 0)$).
3. The element $\sup(x, -x)$ is called the absolute value of and is written $|x|$.

43 The set of rational numbers.

43.1 The set Z of integers

The set Z of integers is the set obtained when we embed N, the natural numbers, in an *additive group.* N is then isomorphic with a subset of Z and we identify it with this subset. The *multiplication* defined on N is extended to Z by the following process.

An element of Z is the equivalence class of a pair (a, a') of natural numbers, defined by the equivalence relation

$$(a, a') \sim (b, b') \Leftrightarrow a + b' = a' + b$$

on $N \times N$.

If m and n are the two elements of Z defined by (a, a') and (b, b') respectively, we put

$$mn = \text{the class of } (ab + a'b',\ ab' + a'b).$$

$mn \in Z$ is independent of the elements defining m and n.
This multiplication law, together with the addition already defined, makes Z a commutative *ring* with a unit, and the law induced on N by the multiplication on Z is the original multiplication on N.

Finally, we extend to Z the order relation of N by saying that the element m of Z defined by (a, a') is greater than 0 if $a > a'$, and defining $m \geqslant n$ to mean $m - n < 0$ or $m = n$.
This relation must, of course, be shown to be an order relation, compatible with addition, and with multiplication by positive elements, and that Z, with this order relation, is *totally ordered*.

43.2 Definitions and properties of the set Q of rationals

Q, the field of fractions of the ring Z, is called the *set of rational numbers*.
The subset of Q that we obtain when we embed N in a group for multiplication and to which we then adjoin 0, is denoted by Q_+ and called the *set of positive rationals*. We write $x \in Q_+$ or $x \geqslant 0$. A *total order* relation is defined on Q by putting

$$x \leqslant y \Leftrightarrow y - x \in Q_+$$

for $x, y \in Q$.
This relation is compatible with the addition, so that for $x, y, z \in Q$

$$x \leqslant y \Rightarrow x + z \leqslant y + z,$$

and with multiplication by positive elements, so that

$$x \leqslant y \Rightarrow xz \leqslant yz,$$

for $x, y \in Q$, and $z \in Q_+$.
$x \in Q$ is *strictly positive* of $(x \in Q_+) \wedge (x \neq 0)$, and we then write $x > 0$. $x < 0$ is defined by $(-x) > 0$, and then we say that x is strictly negative.
For any two rational numbers a, b (and we may suppose $a \leqslant b$), there is a rational lying between a and b, for example $(a + b)/2$.
If $0 < a < b$ we can find an integer $n > 0$ such that $na > b$. In particular, for every rational $a > 0$ there exists an integer n such that $na > 1$ or $1/n < a$.
An *absolute value* can be defined on Q since it is a totally ordered Abelian group and so a Riesz group. We put

$$|x| = x \text{ if } x \geqslant 0,\ |x| = -x \text{ if } x \leqslant 0.$$

This absolute value has the two fundamental properties and also has a property related to the multiplication law on Q: for all $x, y \in Q$, $|xy| = |x||y|$.

The absolute value thus has the following properties, true for all
$x, y, z \in Q$

$$|x| = 0 \Leftrightarrow x = 0,$$

$$|x + y| \leqslant |x| + |y|,$$

$$|xy| = |x||y|.$$

Also

$$||x| - |y|| \leqslant |x - y|,$$

and we observe that if $a \in Q_+$, $|x| \leqslant a$ means that $-a \leqslant x \leqslant a$, $|x| \geqslant a$ implies $x \geqslant a$ or $x \leqslant -a$, $|x| < a$ means $-a < x < a$, $|x| > a$ means $x > a$ or $x < -a$.

43.3 Topology on Q

43.3.1 Intervals

For two rationals a, b such that $a \leqslant b$, the set of rationals x such that $a < x < b$ is called the *open interval with end points* a, b. It is denoted $]a, b[$.
a is called the left extremity or end-point, and b the right extremity or end-point. We shall always tale $]a, b[$ to be an open interval such that $a \leqslant b$, so that the first letter represents the left-hand end-point.
If $a = b$

$$]a, b[= \emptyset$$

If $a < b$

$$]a, b[\neq \emptyset$$

since $(a + b)/2 \in]a, b[$.
The point $(a + b)/2$ is called the *mid-point* of the interval $]a, b[$.
Since Q is an additive group, every non-empty open interval $]a, b[$ is obtained from $](a - b)/2, (b - a)/2[$ by the translation $(a + b)/2$. We shall write

$$]a, b[=](a - b)/2, (b - a)/2[+(a + b)/2.$$

The interval $](a - b)/2, (b - a)/2[$ has mid-point 0, the unit element of Q for addition. If we put $r = (b - a)$ the interval can also be written $] - r/2, r/2[$.
Finally, the set of $x \in Q$ such that $a \leqslant x \leqslant b$ is called a *closed interval* and is denoted by $[a, b]$. The extremities of a closed interval belong to the interval and if $a = b$, $[a, b]$ reduces to the single point a, and so is non-empty.

43.3.2 Basis for a topology, basis of open sets

We take the set $\mathscr{T}$ of open intervals of Q as a basis for a topology (basis for the open sets).
It is easy to verify that $\mathscr{T}$ is a fundamental family, that the set of open intervals covers Q, and that if

$$x \in]a,b[\bigcap]a'b'[= \emptyset$$

there exists an $]a'',b''[$ containing x and contained in the intersection of two intervals. To prove this, we observe that is

$$x \in]a,b[\bigcap]a',b'[\neq \emptyset$$

we have $x < b$ and $x < b'$, and that $a < x$ and $a' < x$, so that it suffices to take

$$a'' = sup(a,a'),\ b'' = inf(b,b')$$

If we define the set of open intervals containing 0 to be $\mathscr{B}(0)$, a basis of open neighborhoods for 0, and define $\mathscr{B}(x) = \mathscr{B}(0) + x$ for every $x \in Q$.
We may also take for $\mathscr{B}(0)$ the set of open intervals having 0 as midpoint, or the open intervals $]-1/n, 1/n[$, where n varies over N.
Using the property that for every rational $a > 0$ we can find as integer n such that $1/n < a$, it is easy to see that this later topology is equivalent to that considered.
Q being countable, the topology on Q is defined by a *countable basis.*

43.3.3 Topological properties

(1) **Q is a Hausdorff space.** In fact if $x, x' \in Q$ and $x \neq x'$, if we put $r = |x' - x|$ then

$$]x - r/2, x + r/2[\in \mathscr{B}(x),\ ]x' - r/2, x' + r/2[\in \mathscr{B}(x'),$$

are these two disjoint intervals.

(2) **The closure of an open interval $]a,b[$ is the closed interval $[a,b]$.** a and b are adherent points to $]a,b[$ since every open interval containing a or b has a non-empty intersection with $]a,b[$.

On the other hand, if $c \notin]a,b[$ there is a open interval containing c which does not meet $]a,b[$, so that c is not adherent to $]a,b[$ and the closure of $]a,b[$ is therefore $[a,b]$.

(3) **Q is not locally compact and so, *a fortiori* is not compact.**

By definition, Q will be locally compact if for every $x \in Q$ there is an $X \in \mathscr{B}(x)$ such that $\overline{X}$ is compact. Next week we will prove the theorem, that $\overline{X}$ is compact if and only if every filter on $\overline{X}$ has an adherent point.

The rest of the proof is left as a homework.

43.3.4 Sequences

Convergent sequences were defined in paragraph 28.3.2. We recall the properties of convergent sequences of rationals, properties arising from those of Q.

1. A sequence (x_n) converges to a rational x_0 if and only if $(x_n - x_0)$ converges to 0.
2. If (x_n) converges to x_0, $(|x_n|)$ converges to $|x_0|$.
3. (x_n) converges to 0 if and only if $(|x_n|)$ converges to 0.
4. If (x_n) converges to x_0 and (y_n) to y_0, $(x_n + y_n)$ converges to $x_0 + y_0$.
5. If (x_n) converges to x_0, the double sequence $(x_p + x_q)_{p,q}$ converges to 0.

The last property leads to the definition of Cauchy sequences.

Definition 43.1. A sequence (x_n) of rational numbers is called a Cauchy sequence if

$$\lim_{p,q\to\infty}(x_p - x_q) = 0.$$

The Cauchy sequences are more general than convergent sequences, in sense that there are Cauchy sequences in Q which do not converge in Q. (As a homework assignment, please find an example of such sequence).

Two following propositions are evident:

Proposition 43.1. Every convergent sequence is a Cauchy sequence

Proposition 43.2. Every sub-sequence of a Cauchy sequence is a Cauchy sequence

Proposition 43.3. If a Cauchy sequence (x_n) has a convergent subsequence the (x_n) is convergent.

Proof. We may suppose that the sub-sequence (x_{n_k}) converges to 0 (since for every sequence $(x_n - x_0)$ converges to 0). Let $I =]-\varepsilon, \varepsilon[$ be an open interval with mid-point 0. Since (x_n) is a Cauchy sequence, for p and q greater than some integer P we have

$$|x_p - x_q| < \varepsilon/2;$$

and so $|x_p - x_{n_k}| < \varepsilon/2$ if p and n_k are both greater than P. But since (x_{n_k}) converges to 0 we have $|x_{n_k}| < \varepsilon/2$ for all n_k greater than some integer P'. If P'' is the greater of P and P' then

$$|x_p| = |x_p - x_{n_k} + x_{n_k}| \leqslant |x_p - x_{n_k}| + |x_{n_k}| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $p \geqslant P''$.

□

Proposition 43.4. If (x_n) and (y_n) are Cauchy sequences then $(x_n + y_n)$ is a Cauchy sequence and so is $(-x_n)$.

Proof. This follows from

$$|x_p + y_p - x_q - y_q| \leqslant |x_p - x_q| + |y_p - y_q|.$$

$\square$

Proposition 43.5. If (x_n) is a Cauchy sequence so is $(|x_n|)$.

Proof. To prove this we use the inequality

$$||x_p| - |x_q|| \leqslant |x_p - x_q|$$

$\square$

Proposition 43.6. If (x_n) is a Cauchy sequence and if the sequence (y_n) is such that $(x_n - y_n)$ tends to 0 then (y_n) is a Cauchy sequence.

Proof. This follows from

$$\begin{aligned} |y_p - y_q| &= |y_p - x_p + x_p - x_q + x_q - y_q| \\ &\leqslant |y_p - x_p| + |x_p - x_q| + |x_q - y_q| \end{aligned}$$

$\square$

Proposition 43.7. If (x_n) is a Cauchy sequence there exists a rational $M > 0$ such that $|x_n| \leqslant M$ for all n.

Proof. There exists P such that $|x_p - x_q| < \varepsilon$ for $p, q > P$. Thus if $p \geqslant P$, $|x_p - x_P| < \varepsilon$ and so $|x_p| < |x_P| + \varepsilon$. We may now take

$$M = \sup(|x_1|, |x_2|, \dots, |x_{P-1}|, |x_P| + \varepsilon)$$

$\square$

This proposition may be also stated: every Cauchy sequence is *bounded*.

Proposition 43.8. If (x_n) is a Cauchy sequence which does not converge to 0 then there exists a strictly positive rational a such that, except for possible a final set of values of n, one, and only one of the inequalities,

$$x_n < -a, \ x_n > a,$$

is true. It follows that $|x_n| > a > 0$ except for a finite set of values of n.

Proof. In fact if (x_n) does not tend to 0 there exists $b > 0$ such that for an infinity of values of n, $|x_n| > b$. Let $\varepsilon = b/2$, then there exists $n_1 \geqslant n_0$ such that $|x_{n_1}| > 2\varepsilon = b$, and for $p \geqslant n_0$, $q \geqslant n_0$ we have

$$|x_p - x_q| < b.$$

It follows that for $n \geqslant n_1$:

$$|x_n - x_{n_1}| < \varepsilon$$

or

$$x_{n_1} - \varepsilon < x_n < x_{n_1} + \varepsilon$$

Since $|x_{n_1}| > 2\varepsilon$, the numbers $x_{n_1} - \varepsilon$ and $x_{n_1} + \varepsilon$ are either both positive or both negative. If $x_{n_1} > 0$, putting $x_{n_1} - \varepsilon = a > \varepsilon$ we have, for $n > n_0$

$$x_n > a.$$

If $x_{n_1} < 0$, putting $x_{n_1} + \varepsilon = -a > \varepsilon$ we have $a > \varepsilon$, and for $n > n_0$

$$x_n < -a.$$

□

44 The construction of R and its fundamental properties

44.1 Definition of R

Let Γ be the set of Cauchy sequences in Q and let $\mathscr{R}$ be relation between two Cauchy sequences (x_n) and (x'_n) in Q, defined by

$$(x_n)\mathscr{R}(x'_n) \Leftrightarrow \lim_{n\to\infty}(x_n - x'_n) = 0.$$

$\mathscr{R}$ is an *equivalence relation*. For, writing the relation $\sim$, for brevity, we have, for arbitrary Cauchy sequences (x_n), (x'_n), (x''_n), in Q:

1. $(x_n) \sim (x_n)$.
2. $(x_n) \sim (x'_n) \Rightarrow (x'_n) \sim (x_n)$, since

$$\lim_{n\to\infty}(x_n - x'_n) = 0 \Leftrightarrow \lim_{n\to\infty}|x_n - x'_n| = 0$$

3. $(x_n) \sim (x'_n)$ and $(x'_n) \sim (x''_n) \Rightarrow (x_n) \sim (x''_n)$, since

$$x_n - x''_n = x_n - x'_n + x'_n - x''_n.$$

Definition 44.1. The set $R = \Gamma/\mathscr{R}$ is called the set of real numbers.

R is this the set of equivalence classes of Γ modulo $\mathscr{R}$.
We now define an additive group law on R, an order relation, and a topology, and show that Q is isomorphic to a subset of R, and finally we give the fundamental properties of the set R with these structures.
Notation. We shall use Greek letters to denote the elements of R. Thus ξ will be the equivalence class of the Cauchy sequence (x_n), ξ' that of (x'_n), η that (y_n), etc., and we shall write $\xi = cl(x_n)$.

44.1.1 Formal identification of Q with a subset of R

To each rational number x we assign in R the class ρ defined by the Cauchy sequence (x_n) such that $x_n = x$ for all n.
The mapping $x \to \rho$ of Q into R is a biuniform mapping of Q onto a subset of R (the set of the ρ). For if $\rho = cl(x_n)$, $\rho' = cl(x'_n)$, $\rho = \rho'$ implies that $(x) \sim (x')$, or $\lim(x - x') = 0$, so that $x = x'$.
We can now identify Q formally, with a subset of R, but we shall show later that the biuniform mapping so defined is an isomorphism for addition and multiplication, etc.

44.2 Addition, order, absolute value in R

44.2.1 Addition

We have seen that if (x_n) and (y_n) are Cauchy sequences in Q, so is $(x_n + y_n)$. If $(x'_n) \sim (x_n)$, $(x_n + y'_n) \sim (x_n + y_n)$. We may therefore give the following definitions for elements ξ, ξ', ... of R:

$$\begin{aligned} \xi + \xi' &= cl(x_n + x'_n), \\ 0 &= cl(0), \\ -\xi &= cl(-x_n). \end{aligned}$$

It is easy to see that the internal law so defined on R is an Abelian group law, and induces the initial addition law on Q. So we have

Proposition 44.1. The set R is an Abelian group for addition.

44.2.2 Order

The sequences equivalent to (0) and the sequences converging to 0 clearly coincide. Let ξ be an element of R. If $\xi \neq 0$ it contains only Cauchy sequences (x_n) which are not equivalent to 0. It follows by Proposition (43.8), that there is a rational $a > 0$ such that $x_n > a$ or $x_n < -a$ for all but possibly a finite number of values of n, and this is true for one of the two inequalities. Suppose for example, that $x_n > a > 0$ except for a finite set of values of n, and let (y_n) be another

Cauchy sequence belonging to ξ, and so equivalent to (x_n). Since (y_n) is not equivalent to 0 there is a rational $b > 0$ such that $y_n > b$ or $y_n < -b$ except for a finite set of values of n.
Then if $x_n > a > 0$ we must have $y_n > b > 0$ for it, except for a finite set of values of n, $x_n > a > 0$ and $y_n < -b < 0$ we should have $x_n - y_n > a + b$, which contradicts the hypothesis $(x_n) \sim (y_n)$, i.e.

$$\lim_{n\to\infty} (x_n - y_n) = 0$$

This observation justifies the following definition:

Definition 44.2. An element ξ of R will be called strictly positive, and we shall write $\xi > 0$, if ξ contains a Cauchy sequence (x_n) such that $x_n > a > 0$, where a is a rational, except perhaps for a finite set of values of n. $\xi \in R$ will be called positive, and we shall write $\xi \geqslant 0$, if $\xi > 0$ or $\xi = 0$.

We observe that ξ will be positive if it contains a Cauchy sequence (x_n) all of whose elements are positive. For either (x_n) is not equivalent to 0 and then $\xi > 0$, or it is equivalent to 0 and $\xi = 0$. Conversely, if $\xi \geqslant 0$ it contains a Cauchy sequence (x_n) all of whose elements are greater than or equal to 0. In fact either $\xi > 0$ and then the desired property is evident (omitting, if necessary, a finite set of terms of the sequence defining ξ), or $\xi = 0$. If $\xi = 0$ it is defined by a sequence (x_n) which converges to 0. If $x_n \geqslant 0$ the argument is complete: if $x_n \leqslant 0$, $(-x_n) \in \xi$, and if x_n is positive for an infinity of values n_k of n, the sequence $y_k = x_{n_k}$ is a positive sequence, converging to zero.
We define $\xi > \xi'$ by $\xi - \xi' > 0$, $\xi \geqslant \xi'$ by $\xi - \xi' \geqslant 0$, $\xi < 0$ by $-\xi > 0$, $\xi \leqslant 0$ by $-\xi \geqslant 0$.
We can now verify that this order relation is a total order on R and is compatible with addition in R.
Finally, our order relation induces the initial order relation on Q, since two distinct real numbers are separated by a rational, (this the set of real numbers lying strictly between two real numbers is non-empty), and R satisfies *Archimedes' axiom*, i.e. if ξ and ξ' are positive there is an integer $n > 0$ such that $\xi' < n\xi$.

44.2.3 Absolute value

We define an absolute value on R by putting $|\xi| = \sup(\xi, -\xi)$ or, if (x_n) is a Cauchy sequence defining ξ, by putting $|\xi| = cl(|x_n|)$ (by Proposition (43.5)).
Note that if (x_n) and (y_n) are two Cauchy sequences in Q, the inequality

$$||x_n| - |y_n|| \geqslant |x_n - y_n|$$

implies that

$$(x_n) \sim (y_n) \Rightarrow (|x_n|) \sim (|y_n|)$$

This remark justifies the definition $|\xi| = cl(|x_n|)$, since if we replace (x_n) by $(y_n) \sim (x_n)$, $|\xi|$ is unaltered.
We recall the properties of the absolute value:

1. If $\xi = x \in Q$, the absolute value of x defined in R is the same as that defined in Q.
2. For arbitrary real ξ, ξ'
   (a) $|\xi| = 0 \Leftrightarrow \xi = 0$,
   (b) $|\xi + \xi'| \leqslant |\xi| + |\xi'|$,

   and from (b) we deduce
$$||\xi| - |\xi'|| \leqslant |\xi - \xi'|$$

44.3 The field R

The definition of the topology on R does not make use of multiplication. That is why we have been able to delay consideration of multiplication until now. We shall later point out the topological properties of R which depend on the existence of a multiplication.

44.3.1 Multiplication in R

If (x_n) by (y_n) are Cauchy sequences in Q, so is $(x_n y_n)$ since:

$$|x_p y_p - x_q y_q| = |x_p(y_p - y_q) + y_p(x_p - x_q)| \leqslant M(|y_p - y_q| + |x_p - x_q|)$$

by Proposition (43.7).
Also, if $(x_n) \sim (x'_n)$, we have
$$(x_n y_n) \sim (x'_n y_n),$$
since
$$|x_n y_n - x'_n y_n| \leqslant M|x_n - x'_n|.$$
This shows that if $\xi = cl(x_n)$, $\xi' = cl(x'_n)$ we may put

$$\xi\xi' = cl(x_n x'_n)$$

and this enables us to define a second internal law on R, multiplication, which is both *associative and commutative.*
The element of R defined by the Cauchy sequence (x_n) where $x_n = 1$ for all n is the unit element. We shall continue to denote it by 1.
It is easy to verify that multiplication is *distributive* with respect to addition.
If $\xi \neq 0$ and (x_n) is a Cauchy sequence defining ξ, there is a rational $a > 0$ such that $|x_n| > a$ for $n \geqslant n_0$.
Since
$$|1/x_p - 1/x_q| = |x_p - x_q|/|x_p x_q| < |x_p - x_q|/a^2,$$

the sequence $(1/x_n)$, where $n \geqslant n_0$, is a Cauchy sequence. Also, if $(x'_n) \sim (x_n)$, omitting if necessary a finite set of terms, we have

$$(1/x'_n) \sim (1/x_n).$$

We then put $1/\xi = \xi^{-1} = cl(1/x_n)$ and have

$$\xi\xi^{-1} = 1$$

The multiplication induced by R on Q is the multiplication initially defined in Q.
These results and definitions cane be summarized:

Theorem 44.1. *R is a commutative field.*

44.3.2 Properties of order and absolute values

We recall the two following properties:

1. $\xi \geqslant \xi'$ and $\xi'' \geqslant 0 \Rightarrow \xi\xi'' \geqslant \xi'\xi''$,
2. $|\xi\xi'| = |\xi||\xi'|$.

44.4 The topology on R. The two fundamental properties.

44.4.1 The topology

For each $\xi \in R$ we define the basis of open neighborhoods, $\mathscr{B}(\xi)$ to be the set of open intervals $]\alpha, \beta[$ containing ξ.
In the homework please prove the last statement, i.e. demonstrate that the set of open intervals $]\alpha, \beta[$ containing ξ is the basis of open neighborhoods for ξ.
We can obtain an equivalent topology by taking as a basis for the open neighborhoods of ξ the open intervals with ξ as mid-point (Proof of this statement is left as a homework).
We can now talk of convergent sequences and convergent double sequences in the topological product $R^2 = R \times R$, and so define Cauchy sequences by using the additional law.

Definition 44.3. We shell say that the sequence (ξ_p) of real numbers is a Cauchy sequence in R if the double sequence $(\xi_p - \xi_q)_{p,q}$ converges to 0 in R^2.

The handling of the topology on R is simplified by the use of the absolute value.
Just as in the case of the algebraic laws and the order relation we have verified that they induce on Q the initial laws and relations, so with the topology on R we must verify that it induces the initial topology on Q. The verification of this statement is left as a homework, i.e. you should prove:

Proposition 44.2. The topology defined on R by the open intervals induces on Q a topology equivalent to that initially defined there.

This property shows that the convergent sequences of rationals in Q (or $Q \times Q$) are the same whether we consider the initial topology on Q or that induced by R.
We now come to two fundamental properties of R:

Theorem 44.2. *Q is dense in R*

Proof. We have to show that for every $\xi \in R$, every open interval containing ξ contains a rational. Now ξ is the equivalence class of a Cauchy sequence (x_n) of rationals. If x_p is a member of the sequence, $(\xi - x_p)$ is the equivalence class of the sequence

$$(x_n - x_p)_{n \in N}$$

and since (x_n) is a Cauchy sequence (in Q and in R), for n and p greater than some suitably chosen integer, we have

$$|x_n - x_p| < \varepsilon.$$

For such p, $|x_n - x_p| < \varepsilon$ for sufficiently large n, which shows that

$$x_p \in]\xi - \varepsilon, \xi + \varepsilon[$$

□

Theorem 44.3. *A sequence (ξ_n) of real numbers converges if and only if it is a Cauchy sequence.*

Proof. If (ξ_n) is convergent there exists a ξ such that, given $\varepsilon > 0$, $\xi_n \in]\xi - \varepsilon/2, \xi + \varepsilon/2[$ for $n > P(\varepsilon)$, or $|\xi_n - \xi| < \varepsilon/2$, which by the triangle inequality, implies that

$$|\xi_p - \xi_q| < \varepsilon \text{ for } p, q > P(\varepsilon)$$

Conversely, suppose that for every $\varepsilon > 0$ there exists $P(\varepsilon)$ such that $|\xi_p - \xi_q| < \varepsilon$ for $p, q > P(\varepsilon)$. Let (ε_p) be a sequence converging to zero.
By Theorem (44.2), to each ξ_p we can assign a rational x_p such that $|\xi_p - x_p| < \varepsilon_p$. We then have

$$|x_p - x_q| \leqslant \varepsilon_p + \varepsilon_q + |\varepsilon_p - \varepsilon_q|$$

so that $\lim_{p,q\to\infty} |x_p - x_q| = 0$. The sequence (x_n) being a Cauchy sequence of rationals, it defines a $\xi \in R$ and by Theorem (44.2)

$$\xi = \lim_{p\to\infty} \xi_p$$

□

Instead of saying "every Cauchy sequence in R is convergent" we say "R is complete", then we have the following result:

Theorem 44.4. *R is complete.*

The notion of a complete space (which we shall come across later when we come to study metric spaces) is a fundamental concept. Its importance lies in the fact that, if we want to know whether a sequence is convergent, if we already know that the space is complete it is no longer necessary to find the limit of the sequence; it suffices to show that $\xi_p - \xi_q$ tends to zero.

44.4.2 Topological properties of the totally ordered field R

From now on we shall no longer distinguish between the symbols which denote rationals and those which denote real numbers.
Appealing to the definition of continuity we obtain immediately the following properties of the field R.

1. The map $x \to -x$ of R into R is continuous.
2. The map $(x, y) \to (x + y)$ of $R \times R$ into R is continuous.
3. The map $(x, y) \to xy$ of $R \times R$ into R is continuous.
4. The map $x \to x^{-1} = 1/x$ of R^* (the set R, from which 0 has been omitted) into R^* is continuous.

Properties (1) and (2) are those used to define a *topological group*, written additively.
The properties (1), (2) and (3) define a *topological ring.*
(1), (2), (3) and (4) define a *topological field.*
In the case of sequences of real numbers, these properties correspond to the elementary results known as "the theorems of limits of sequences".
We have the following important property relating to topology and order:

Proposition 44.3. If a sequence (x_n) of real numbers is convergent and if its terms are all non-negative (non-positive) its limit x_0 is non-negative (non-positive).

Proof. Suppose $x_n \geqslant 0$ and $x_0 = \lim x_n$. If we had $x_0 < 0$ then for $\varepsilon = |x_0|/2$ we should have $x_0 + \varepsilon = x_0 + |x_0|/2 < 0$, and except for a finite set of values of n, $x_0 - \varepsilon < x_n < x_0 + \varepsilon < 0$, which contradicts the hypothesis "$x_n \geqslant 0$ for all n". □

In a similar way it is easy to prove that:

Proposition 44.4. The set of positive real numbers is closed

45 The real line

The additive group of real numbers, with the topology defined in section (44) is called the *real line.*

45.1 Properties of the topology of R

45.1.1 The intervals

If a and b are elements of R we recall that $]a, b[$ is called an open interval (in this notation we always suppose that $a \leqslant b$, and that $]a, b[$ is the set of $x \in R$ such that $a < x < b$). An open interval is empty only if $a = b$.
The set of numbers $x > a$ $(< a)$ is denoted by $]a, +\infty[$, $(]-\infty, a[)$, and is again called an interval.
These intervals are called *open sets*, for if p is the first integer greater than a:

$$]a, +\infty[=]a, p+1[\cup X,$$

where

$$X = \bigcup_{k=p}^{\infty}]k, k+2[,$$

and a union of open intervals is an open set.
The interval $]a, +\infty[$ $(]-\infty, a[)$ is also called the *half-line with left-hand end-point* a (*right-hand end-point* a). We write $R =]-\infty, +\infty[$.
The open intervals $]a, b[$ are called bounded.
$[a, b]$, $([a, +\infty[,]-\infty, a])$, denotes the set of $x \in R$ such that

$$a \leqslant x \leqslant b \ \ (a \leqslant x, x \leqslant a).$$

These sets are called *closed intervals.* A closed interval is never empty.
The closed intervals are closed sets since their complements are open.
We shell use the word interval to denote the set of real numbers greater than a, less than a, or contained between a and b, whether or not the points a and b belong to the set.
A subset A of R is called *bounded* if it is contained in a bounded interval. If A is bounded there exist $a, b \in R$ such that for every $x \in A$, $a \leqslant x \leqslant b$. This comes to the same thing as saying that there is a $c > 0$ such that for every $x \in A$, $|x| \leqslant c$.

45.1.2 Basis for the topology

The open intervals form a basis for the topology.
If E is a set dense in R, for each $x \in R$ consider the open intervals $]\alpha, \beta[$, where $\alpha, \beta \in E$. If $]a, b[$ is an open interval containing x since E is dense in R, between a and x there is an $\alpha \in E$ such that $a \leqslant \alpha \leqslant x$, and between x and b a $\beta \in E$ such that $n < \beta \leqslant b$. this proves that every $]a, b[$ containing x contains an $]\alpha, \beta[$ containing x.
It follows that the open intervals whose end points belong to E form a basis for the topology of R.
In particular let $E = Q$. Since Q is also countable, the set of $]\alpha, \beta[$ where $\alpha, \beta \in Q$ is countable and:

Proposition 45.1. The topology of R has a countable basis.

It is also clear that

Proposition 45.2. R is a Hausdorff space

45.1.3 Open sets and closed sets

Proposition 45.3. Every open set of R is countable union of closed sets, and every closed set is a countable intersection of open sets.

Proof. Let O be a non-empty open set in R. To each $a \in R$ we assign the interval $]a-1/n, a+1/n[$, where $n \in N$. Consider in O the set O_n of points $x \in O$ such that

$$]x - 1/n, x + 1/n[\subset O$$

The sets O_n are not all empty since, O being open, for every $x \in O$, $]x - 1/n, x + 1/n[\subset O$ if n is sufficiently large.
We have $O_n \subset O_{n+1}$ for all n.
Now let y be a point adherent to O_n. If $y \in \overline{O}_n$, $]y - 1/n, y + 1/n[$ contains a point $x \in O_n$ and $]x - 1/n, x + 1/n[$ contains y. Thus, by definition of O_n, $y \in O$. It follows that $\overline{O}_n \subset O$ for all n and so

$$\cup \overline{O}_n \subset O.$$

Now every point $x \in O$ belongs to at least one O_n since O being open, for every $x \in O$ there is an open interval $]x - 1/n, x + 1/n[$ containing x and contained in O. It follows that

$$O \subset \cup O_n \subset \overline{O}_n$$

and so $O = \overline{O}_n$. Since the closure is a closed set and the complement of an open set is closed, the second part of proposition is follows. □

45.1.4 Closure, point of accumulation

Let A be a non-empty subset of R. If there is a sequence (x_n) of points of A which converges to a point x, $x \in \overline{A}$. Conversely, if $x \in \overline{A}$, for all $n \in N$, $]x - 1/n, x + 1/n[$ contains a point of A which we shall denote by x_n. Since the intervals $]x - 1/n, x + 1/n[$ are decreasing, the sequence converges to x.

Proposition 45.4. A point x is adherent to a non-empty subset A of R if and only if there is a sequence of points of A which converges to x.

The concept of point of accumulation if often useful.
This concept (which can be introduced in general topological space) arises from the classification of adherent points into two categories. Among the points x adherent to a set E there will be some with the property that every sufficiently small open set containing x contains only a single point of A, x itself. This implies that such a point belongs to A. It is called an *isolated point.*
If x is adherent to A but is not isolated then every open interval containing x contains a point of A (other than x, if $x \in A$). There is an open interval containing a point x_1 of A, then, in a second open interval $]x - a_1, x + a_2[$ not containing x_1 (which is possible since $x \neq x_1$) we choose a point $x_2 \in A$, etc. In this way we may construct a sequence (x_n) whose elements belong to A and are all different, and which converges to x. The converse is obvious. A point x with this property is called a point of accumulation (note that only the subsets of A having an infinity of elements can have points of accumulation). We now formulate the following:

Definition 45.1. A point x is called a point of accumulation of a subset A of R containing infinitely many points if every open interval containing x contains a point of A other than x
The set of points of accumulation of A is called the derived set of A.

The closure of a set A can now be seen to be the union of the isolated points of A and the points of accumulation of A.