MA651 Topology. Lecture 8. Compactness 1.

This text is based on the following books:

- *"Topology" by James Dugundgji*
- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*
- *"Linear Algebra and Analysis" by Marc Zamansky*
- *"General topology I", A.V. Arkhangel'skii and L.S. Pontryagin (Eds)*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

46 Compactness

Compactness is a topological concept which was originally inspired by properties of point sets in E^n. It was recognized quite early that certain kinds of sets had advantages over others in calculus. For example, a function continuous on a closed and bounded set achieves a maximum and a minimum.

When topology was in its formative stages, there was no obvious was to generalize "closed and bounded". The prototype of the compactness notion is the following property of a segment (Lebesgue, 1903): any open covering of a segment $[a,b] = \{x \in R \mid a \leqslant x \leqslant b\}$ of the real line R contains a finite subcovering. Later Borel generalized it for a finite dimensional Euclidean space E^n, this theorem is learned in calculus as the Heine-Borel Theorem. A set A in E^n is closed and bounded exactly when every cover of A by open sets can be reduced to a finite cover. That is, if $A \subset \bigcup_{\alpha \in B} t_\alpha$, where the sets t_α are open, then it is possible to select finitely many of the t_α's such that $A \subset t_{\alpha_1} \cup t_{\alpha_2} \cup \cdots \cup t_{\alpha_n}$. This notion of reducibility of open covers generalizes to any space. A set with the property that each open cover has a finite reduction will be called compact.

The word "compact" has an interesting history. Around 1906, Fréchet used compact to mean that every infinite subset of A has a cluster point in A. This was probably motivated by the Boltzano-Weierstrass Theorem. Alexandroff and Urysohn, about 1924, used the notion of reduction of open covers, but called it bicompactness. Bourbaki dropped the prefix "bi", but restricted "himself" to Hausdorff spaces. When examples appeared in differential geometry of non-Hausdorff spaces having the reducibility property for open covers, Bourbaki labeled such spaces quasi-compact. We will use word compact for any arbitrary topological space.

Definition 46.1. An open cover of a topological space $(X, \mathscr{T})$ is a family of open sets Γ such that $\cup\Gamma = X$. If Ω is a subfamily of sets $\Omega \subset \Gamma$ and $\cup\Omega = X$ then Ω is called a *subcover* of the cover Γ.

Definition 46.2. *(Axiom of Borel-Lebesgue.)* A topological space $(X, \mathscr{T})$ is said to be *compact* (or $\mathscr{T}$-compact) if every open set cover Γ of X has a finite subcover μ.

Example 46.1. *E^n is not compact. This is easiest to see when $n = 1$, where the open intervals $]m-1, m+1[$, as m takes on all integer values, form an open cover which cannot be reduced by as much as one set. A similar construction works in E^n. Take, for example, the spheres $B_{\varrho_m}((x_1, \dots, x_n), \sqrt{m})$, where each coordinate x_i of $(x_1, \dots, x_n)$ is an integer.*

Example 46.2. *Any indiscrete space is compact.*

Example 46.3. *A discrete space is compact if and only if X is finite.*

Example 46.4. *Let $X = Z^+$, and let $\mathscr{T}$ consists of X, $\emptyset$ and all sets $[1, n] \cap Z^+$, where $n \in Z^+$. Then, X is not $\mathscr{T}$-compact.*

We can speak of a subset A of X as being $\mathscr{T}$-compact or non $\mathscr{T}$-compact by referring A to the subspace topology induced by $\mathscr{T}$.

Definition 46.3. Let $A \subset X$, then A is $\mathscr{T}$-compact if and only if A is $\mathscr{T}_A$-compact.

In practice, compactness of A can always be decided without actually going into the relative topology. We call a collection of $\mathscr{T}$-open sets a cover of A if A is a subset of their union (but not necessarily equal to their union). Compactness of A can then be tested by looking at $\mathscr{T}$-covers of A, and $\mathscr{T}_A$-covers never have to be considered.

Example 46.5. *In Example (46.4), a subset A of Z^+ is compact exactly when it is bounded above. That is, the only compact sets are the finite subsets of Z^+. Conversely, each finite subset is compact.*

Example 46.6. *The empty set is a compact subset of any space, as is $\{x_1, \dots x_n\}$ for any finite number of points $x_1, \dots, x_n$ in X.*

The next theorem is extremely useful and provide alternate formulations of the notion of compactness:

Theorem 46.1. *The following are equivalent:*

1. *A topological space X is compact.*
2. *Every family of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is empty.*
3. *Every filter on X has at least one cluster point.*
4. *Every ultrafilter on X is convergent.*

Proof.

(1) $\Rightarrow$ (2) Assume (1). Let $\mathscr{F}$ be non-empty family of closed subsets of X. Suppose that $\cap\mathscr{F} = \emptyset$. We must produce a finite subset of $\mathscr{F}$ whose intersection is empty.

Now, $X = X - \cap\mathscr{F} = \cup\{X - F \mid F \in \mathscr{F}\}$. If $F \in \mathscr{F}$, then $X - F \in \mathscr{T}$, so $\{X - F \mid F \in \mathscr{F}\}$ if an open cover of X. Then, there are finitely many elements $F_1, \ldots, F_n$ of $\mathscr{F}$ such that $X = \bigcup_{i=1}^{n}(X - F_i$. Choose $B = \{F_1, \ldots, F_n\}$, and we then have $B \subset \mathscr{F}$ and $\cap B = \emptyset$.

(2) $\Rightarrow$ (1) Assume (2). Let Γ be an open cover of X. Then, $\Gamma \in \mathscr{T}$ and $X = \cup\Gamma$. Then, $\emptyset = X - \cup\Gamma = \cap\{X - \gamma \mid \gamma \in \Gamma\}$. If $\gamma \in \Gamma$, then $X - \gamma$ is closed. By (2), there is a finite set of elements $\gamma_1, \ldots, \gamma_n$ of Γ such that $\bigcap_{i=1}^{n}(X - \gamma_i) = \emptyset$. Then, $X = \bigcup_{i=1}^{n}\gamma_i$, and so X is a compact space.

Proofs of the statements (3),(4) are left as a homework.

□

Remarks:

1. Sometimes it is said the the family of sets $\mathscr{F}$ has the finite intersection property if $\mathscr{F}$ satisfies (2).
2. Note that, upon careful examination of the proof, the last theorem is just a restatement of DeMorgan's Laws as they relate to compactness.
3. An ultrafilter on a set X is a filter $\mathscr{F}$ such that there is no filter which is strictly finer than $\mathscr{F}$.

47 Compact subsets in R

Before considering compactness in general we consider compact subsets of E^1.

Theorem 47.1. *Let $A \subset E^1$. Then, A is compact if and only if A is closed and bounded.*

Proof. Suppose first that A is compact. Let $\{U_x\} = \{]x-1, x+1[\mid x \in A\}$. Then $\{U_x\}$ is an open cover of A. Since A is compact, there are finitely many points $x_1, \ldots, x_r$ in A such that $A \subset \bigcup_{i=1}^{r}]x_i - 1,\ x_i + 1[$. Then, $A \subset [-\eta, \eta]$, where $\eta = \sup\{|x_i| + 1 \mid 1 \leqslant i \leqslant r\}$, hence is bounded. To show that A is closed, suppose that $y \in \overline{A} - A$. In your homework derive a contradiction by showing that $\{E^1 - [y - \frac{1}{j}, y + \frac{1}{j}] \mid j \in Z^+\}$ has no finite subset which covers A. Conversely, suppose that A is closed and bounded. We consider two cases.

Case 1. $A = [a, b]$ for some real numbers a and b with $a < b$.

Let $\{U_\alpha\}$ be any open covering of $[a, b]$. Let $c = \sup\{x \mid [a, x]$ can be covered by finitely many $U_\alpha\}$; if $c < b$, we derive a contradiction to the definition of c by choosing any $U_\alpha \supset c$, observing that there is a $B(c, r) \subset U_\alpha$, and that since $[a, c-(r/2)]$ can be covered by finitely many sets $U_{\alpha_1}, \cdots, U_{\alpha_n}$, these sets together with U_α are finite open covering of $[a, c+(r/2)]$.

Case 2. A is any closed, bounded subset of E^1.

Since A is bounded, then $A \subset [-\delta, \delta]$ for some real δ. Let $\{U_\alpha\}$ be any open cover of A. Then, $U_\alpha \cup \{E^1 - A\}$ is an open cover of $[-\delta, \delta]$. By case 1, $[-\delta, \delta]$ is compact. Then, there are finitely many sets, say $U_1, \ldots, U_s$ of $\{U_\alpha\}$ such that $[-\delta, \delta] \subset (E^1 - A) \cup \bigcup_{i=1}^{s} U_i$. Then, $A \subset \bigcup_{i=1}^{s} U_i$, so A is compact.

□

48 Compactness as a topological invariant.

We now consider the invariance properties of compactness. As with connectedness, compactness is preserved by continuous surjections, hence also by homeomorphism. This provides us with another topological invariant (to go with connectedness, path connectedness, number of components and number of path components).

Theorem 48.1. *Let $(X, \mathscr{T})$, $(Y, \mathscr{M})$ be topological spaces. Let $f : X \to Y$ be a continuous surjection. Let X be $\mathscr{T}$- compact, then Y is $\mathscr{M}$-compact.*

In another words, if a topological space Y is the image of a compact space X under a continuous map, then Y is compact.

Proof. Let Γ be an $\mathscr{M}$-cover of Y. Then, $\{f^{-1}(\gamma) \mid \gamma \in \Gamma\}$ is a $\mathscr{T}$-cover of X, by continuity of f. Since X if $\mathscr{T}$-compact, there is a finite set of elements $\gamma_1, \ldots, \gamma_n$ of Γ such that $\bigcup_{i=1}^{n} f^{-1}(\gamma_i) = X$. Then, $\bigcup_{i=1}^{n} \gamma_i = Y$, as f is a surjection. □

The following corollary immediately follows:

Corollary 48.1. *Let $(X, \mathscr{T}) \cong (Y, \mathscr{M})$. Then, X is $\mathscr{T}$-compact $\Leftrightarrow$ Y is $\mathscr{M}$-compact.*

Example 48.1. $[0,1] \cup]6,8]$ *is not homeomorphic to* $[0,1] \cup [6,8]$ *in* E^1*, since the latter set is compact by Theorem (47.1), but the former is not. Note that both spaces are disconnected and not path connected, so that testing homeomorphism by the topological invariants that we knew before fails.*

49 Separation properties of compact spaces

Of course, not every subspace of a compact space is compact. For example, $]\frac{1}{2}, \frac{3}{4}[$ is non-compact subspace of compact $[0,1]$. However, a closed subset of a compact space is compact. This will follow easily from a slightly more general statement.

Theorem 49.1. *Let $(X, \mathscr{T})$ be a topological space. Let $A \subset X$ be compact, and $B \subset X$ is closed. Then, $A \cap B$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $A \cap B$. Then $\{U_\alpha\} \cup \{X - B\}$ is an open cover of A, hence can be reduced to a finite subcover, say $\{U_k\} \cup \{X - B\}$, where $\{U_k\} \subset \{U_\alpha\}$. Then, $\{U_k\}$ is an open cover of $A \cap B$, implying that $A \cap B$ is compact. □

Corollary 49.1. *Let $(X, \mathscr{T})$ be a compact topological space. Let $B \subset X$ is closed. Then B is compact.*

Proof. Immediate upon taking $A = X$ in Theorem (49.1). □

The converse of Theorem (49.1) (or of Corollary (49.1)) is not in general true - a compact subset of an arbitrary topological space need not be closed.

Example 49.1. *In the topology $\mathscr{T}$ on Z^+ of Example (46.4), $\{0,1\}$ is compact (as it is finite, see Example (46.5)), but not closed, as $Z^+ - \{0,1\} \notin \mathscr{T}$*

However, when $\mathscr{T}$ is a Hausdorff topology, then every compact subset of X is closed, whether X is compact or not.

Theorem 49.2. *A compact subset A of a Hausdorff space $(X, \mathscr{T})$ is closed; indeed, for each $x \notin A$, there are nonintersecting neighborhoods $U(A)$, $U(x)$.*

Proof. To show that $\mathbf{C}A$ open, we prove that each fixed $x_0 \in \mathbf{C}A$ has a neighborhood lying in $\mathbf{C}A$. For each $a \in A$, find disjoint neighborhoods $U(a)$, $U_a(x_0)$. Since $\{U(a) \cap A \mid a \in A\}$ is an open covering of A, reduce it to a finite covering $U(a_1) \cap A, \ldots, U(a_n) \cap A$; then $U(A) = \bigcap_1^n U(a_i)$ and $U(x_0) = \bigcap_1^n U_{a_i}(x_0)$ are disjoint open sets. □

It is now a trivial statement:

Proposition 49.1. A finite union of compact subsets of a Hausdorff space is compact.

The second statement of Theorem (49.2) is actually a separation property - compact sets can be separated from points in Hausdorff spaces. This suggests that compact Hausdorff spaces might have strong separation properties. In fact, any compact Hausdorff space is normal. Compactness is essential here, as there are Hausdorff spaces which are neither regular nor normal.

Theorem 49.3. *Let $(X, \mathscr{T})$ be compact, Hausdorff space. Then, $\mathscr{T}$ is a normal topology.*

Proof. Let A and B be disjoint closed sets of X. By Corollary (49.1) A and B are compact. If $y \in A$, then by Theorem (49.2) there are disjoint open sets U_y and V_y with $y \in U_y$ and $B \in V_y$. Now, $A \subset \bigcup_{y \in A} U_y$. Since A is compact, there are elements $y_1, \ldots, y_n$ of A such that $A \subset \bigcup_{i=1}^n U_{y_i} \in \mathscr{T}$. Further, $B \subset V_{y_i}$ for $i = 1, \ldots, n$, so $B \subset \bigcup_{i=1}^n V_{y_i} \in \mathscr{T}$. All remains is to observe that

$$\left(\bigcup_{i=1}^n U_{y_i} \right) \cap \left(\bigcup_{i=1}^n V_{y_i} \right) = \emptyset$$

□

A homework assignment is to prove Theorem (49.3) from a "local" point of view using notion of filters and Theorem (46.1).
Just for the record we state the following:

Corollary 49.2. *Let $(X, \mathscr{T})$ be a compact, regular space. Then, $\mathscr{T}$ is a normal topology.*

Proof. Immediate by Theorem (49.3), as a regular space is Hausdorff. □

Theorem 49.4. *Let $(X, \mathscr{T})$ be a compact Hausdorff space. Let F be closed, U be open, and $F \subset U$. Then, there is an open V with $A \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact.*

Proof. By Theorem (49.3), $(X, \mathscr{T})$ is normal. By Proposition (39.1), there is an open V with $A \subset V \subset \overline{V} \subset U$. Since $\overline{V}$ is a closed subset of a compact space, the $\overline{V}$ is compact by Corollary (49.1). □

50 Tychonov Theorem

The following theorem sometimes is considered as the most important theorem in set topology. It says that a product topological space is compact exactly when each coordinate space is compact.

Theorem 50.1. ***(A. Tychonov)***
Let $\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in \mathscr{A}\}$ be any non-empty family of topological spaces. Then $\prod_\alpha X_\alpha$ is compact (with respect to Tychonov's product topology $\mathscr{P}$) if and only if each X_α is $\mathscr{T}_\alpha$-compact.

Proof. Suppose first that $\prod_\alpha X_\alpha$ is $\mathscr{P}$-compact. If $\beta \in \mathscr{A}$, then $p_\beta : \prod_\alpha X_\alpha \to X_\beta$ is a $(\mathscr{P}, \mathscr{T}_\beta)$ continuous surjection, hence X is $\mathscr{T}_\beta$-compact by Theorem (48.1).

Conversely, suppose X_α is $\mathscr{T}_\alpha$-compact for each $\alpha \in \mathscr{A}$. Let $\mathscr{M}$ be an ultrafilter on $\prod_{\alpha\in\mathscr{A}} X_\alpha$. Then using the property that for any $f : X \to Y$ if $\mathscr{K}$ if ultrafilter on X then $f(\mathscr{K})$ is an ultrafilter on Y (Prove this statement in the homework!), we get that $p_\beta(\mathscr{M})$ is an ultrafilter on X_β for each $\beta \in \mathscr{A}$

Since X_β is compact if $\beta \in \mathscr{A}$, then $p_\beta(\mathscr{M})$ is $\mathscr{T}_\beta$-convergent, say to f_β, by Theorem (46.1).

Then we use the property that filter $\mathscr{F}$ is $\mathscr{P}$-convergent (i.e. the filter converge to a some point x of the product space) if and only if $p_\beta(\mathscr{F})$ is $\mathscr{T}_\beta$-convergent to x_β for each $\beta \in \mathscr{A}$ (Prove it in your homework). And $(\mathscr{M})$ is $\mathscr{P}$-convergent to f.

Then by Theorem (46.1) $\prod_{\alpha\in\mathscr{A}} X_\alpha$ is $\mathscr{P}$-compact. □

Theorem 50.2. *Let $A \subset E^n$. Then, A is compact is and only if A is closed and bounded.*

Proof. If A is compact, then A is closed and bounded by an argument similar to that used in the proof of Theorem (47.1). Conversely, suppose that A is closed and bounded. Let $\Omega = \{V_\beta \mid \beta \in \mathscr{B}\}$

be any open cover of A. Since A is bounded, there are real numbers $a_{11}, a_{12}, a_{21}, a_{22}, \ldots, a_{n1}, a_{n2}$ such that

$$A \subset [a_{11}, a_{12}] \times [a_{21}, a_{22}] \times \cdots \times [a_{n1}, a_{n2}]$$

Now $\Omega \cup \{E^n - A\}$ is an open cover of $[a_{11}, a_{12}] \times \cdots \times [a_{n1}, a_{n2}]$, which compact by Theorem (47.1) and Tychonov's Theorem. Hence there is a finite collection $\{V_1, \ldots, V_k\}$ os sets in Ω such that $\{V_1, \ldots, V_k\} \cup \{E^n - A\}$ covers $[a_{11}, a_{12}] \times \cdots \times [a_{n1}, a_{n2}]$. Then, $\{V_1, \ldots, V_k\}$ is a reduction of Ω to a finite cover of A, hence A is compact.

□

51 Alexandroff compactifications

In view of the fact that compact spaces enjoy a number of properties not generally shared by non-compact spaces, it is natural to ask the following question: can a given space be embedded homeomorphically in a compact space? The answer to this is always yes. In fact, there are

many ways of doing it, among them the Wallman, Stone-Čech and Alexandroff compactifications. Today we consider only the Alexandroff method. Let $(X, \mathscr{T})$ is a topological space. Alexandroff's

method is to adjoin a new point, often denoted ∞, to X to form a new set $X \cup \{\infty\}$. A topology $\mathscr{T}'$ is defined on $X \cup \{\infty\}$ by specifying the $\mathscr{T}'$ open set to be $\mathscr{T}$ open sets (which of course do not contain ∞), together with each subset A of $X \cup \{\infty\}$ which contains ∞ and has two additional properties that $A \cap X$ $(= A - \{\infty\})$ is $\mathscr{T}$-open and $X - A$ $(= (X \cup \{\infty\}) - A)$ is $\mathscr{T}$-compact.

The resulting space $(X \cup \{\infty\}, \mathscr{T}')$ is compact, and the map $\varphi : X \to X \cup \{\infty\}$ defined by $\varphi(x) = x$ is an homeomorphism of X into $X \cup \{\infty\}$. Further, if the original space is not compact, then in addition $\overline{\varphi(X)} = X \cup \{\infty\}$ so that ∞ can be approximated as closely as we like (in sense of the topology) by elements of X. Finally, the relative topology $\mathscr{T}'_X$ coincides with $\mathscr{T}$, so that in a sense $(X, \mathscr{T})$ has been left undisturbed. The space $(X \cup \{\infty\}, \mathscr{T}')$ is called Alexandroff one-point compactification of $(X, \mathscr{T})$.

Theorem 51.1. ***Alexandroff***
Suppose that $(X, \mathscr{T})$ is not compact. Let $z \notin X$, $Y = X \cup \{z\}$, and

$$\mathscr{T}' = \mathscr{T} \cup \{A \mid (x \in A \subset Y) \wedge (A \cap X \in \mathscr{T}) \wedge (X - A \textit{ is } \mathscr{T} - \textit{compact})\}$$

Then,

1. *$\mathscr{T}'$ is a topology on Y*
2. *$\mathscr{T}'_X = \mathscr{T}$*
3. *Y is $\mathscr{T}'$-compact*
4. *Let $\varphi(x) : X \to Y$ is a $(\mathscr{T}, \mathscr{T}'_X)$ homeomorphism, and $\overline{\varphi(X)} = Y$*

In practice, the space $(Y, \mathscr{T}')$ of Alexandroff's Theorem can often be realized more concretely through an additional homeomorphism, say $(Y, \mathscr{T}') \cong (P, \mathscr{M})$. In this event, the space $(P, \mathscr{M})$ is also called a one-point compactification of $(X, \mathscr{T})$.

Definition 51.1. Let $(X, \mathscr{T})$ be a non-compact space. Let z, Y and $\mathscr{T}'$ be as in Theorem (51.1). Then, any space homeomorphic to $(Y, \mathscr{T}')$ is an Alexandroff one-point compactification of $(X, \mathscr{T})$.

Note that the assumption that $(X, \mathscr{T})$ is not itself compact is not restrictive, but, of course, in practice, one does not consider compactifications of compact spaces.

Example 51.1. ***The extended real line*** $\overline{R}$. *We know that R is not compact, let us produce a compactification of R. If we consider a bounded open interval $]a, b[$ on R and a homeomorphism between $]a, b[$ and R we are led to adjoin to R two elements $-\infty$ and $+\infty$. For every continuous strictly increasing function f mapping $]a, b[$ (where $a < b$) into R, $-\infty$ is the limit of $f(x)$ when x tends to a on the right, $+\infty$ is the limit of $f(x)$ when x tends to b on the left. We then write $f(a) = -\infty$, $f(b) = +\infty$. The element $-\infty, (+\infty)$ is by definition the limit of every decreasing (increasing) sequence of real numbers which is not minorized (not majorized). The set consisting of R, $-\infty$, $+\infty$ is denoted by $\overline{R}$ and is called* the extended real line. *Every strictly increasing continuous function transforming $]a, b[$ into R preserves order. We therefore agree to put*

$$-\infty < x < +\infty$$

for every $x \in R$. The elements $-\infty$, $+\infty$ are often called infinite real numbers, *whilst those of R are called* finite real numbers. *The topology of R is extended to $\overline{R}$ by taking as a base for the open sets the open intervals $]a, b[$ and the intervals $]a, +\infty[$, $]-\infty, a[$, which are unbounded on the right and left respectively, a and b are finite real numbers. The space $\overline{R}$ is the* compact, *since every covering by open intervals clearly contains a finite covering. Further, every infinite set in $\overline{R}$ has a point of accumulation in $\overline{R}$.*

Example 51.2. *Similarly to the R we can consider higher dimensional Euclidean spaces. For instance, $E^2 \cong S^2 - \{(0, 0, 1)\}$, the stereographic projection being a homeomorphism. A one point compactifiation of E^2 may be then thought of as S^2, the unit sphere in E^3*