MA651 Topology. Lecture 9. Compactness 2.

This text is based on the following books:

- *"Topology" by James Dugundgji*
- *"Fundamental concepts of topology" by Peter O'Neil*
- *"Elements of Mathematics: General Topology" by Nicolas Bourbaki*

I have intentionally made several mistakes in this text. The first homework assignment is to find them.

52 Local compactness

Many of important spaces occurring in analysis are not compact, but instead have a local version of compactness. For example, Euclidean $n-$space is not compact, but each point of E^n has a neighborhood whose closure is compact. Calling a subset A of a topological space $(X, \mathscr{T})$ *relatively* compact whenever its closure $\overline{A}$ is compact, this local property is formalized in:

Definition 52.1. A topological space $(X, \mathscr{T})$ is locally compact if each point has a relatively compact neighborhood.

Example 52.1. *E^n is locally compact; $\overline{B_{\rho_n}(x,1)}$ is compact for each $x \in E^n$. Note that this example also shows that a locally compact subset of a Hausdorff space need not be closed.*

Example 52.2. *Any infinite discrete space is locally compact, but not compact.*

Example 52.3. *The set of rationals in E^1 is not a locally compact space.*

Example 52.4. *Any compact space is locally compact - take X as a neighborhood for every x in Definition (52.1). This is stated as Theorem (52.1) for reference.*

Theorem 52.1. *If $(X, \mathscr{T})$ is a compact topological space, then X is locally compact.*

Example 52.5. *Z^+, with the topology of Example (46.4) is not locally compact and not compact. If $n \in Z^+$, then $Z^+ \cap [1, n]$ is a neighborhood of n which is finite, hence compact, but $\overline{Z^+ \cap [1, n]} = Z^+$*

Equivalent formulations of local compactness are given in the following theorem:

Theorem 52.2. *The following four properties are equivalent:*

1. *X is a locally compact Hausdorff space.*
2. *For each $x \in X$ and each neighborhood $U(x)$, there is a relatively compact open V with $x \in V \subset \overline{V} \subset U$.*
3. *For each compact C and open $U \supset C$, there is a relatively compact open V with $C \subset V \subset \overline{V} \subset U$.*
4. *X has a basis consisting of relatively compact open sets.*

Proof.

(1) $\Rightarrow$ (2) There is some open W with $x \in W \subset \overline{W}$ and $\overline{W}$ compact. Since $\overline{W}$ is therefore a regular space, and $\overline{W} \cap U$ is a neighborhood of x in $\overline{W}$, there is a set G open in $\overline{W}$ such that $x \in G \subset \overline{G_{\overline{W}}} \subset \overline{W} \cap U$. Now $G = E \cap \overline{W}$, where E is open in X, and the desired neighborhood of x in X is $V = E \cap W$.

(2) $\Rightarrow$ (3) For each $c \in C$ find a relatively compact neighborhood $V(c)$ with $\overline{V(c)} \subset U$; since C is compact, finitely many of these neighborhoods cover C, and by Proposition (49.1), this union has compact closure.

(3) $\Rightarrow$ (4) Let $\mathscr{B}$ be the family of all relatively compact open sets in X; since each $x \in X$ is compact, (3) asserts that $\mathscr{B}$ is a basis.

(4) $\Rightarrow$ (1) is trivial.

□

Please note that Theorem (52.2) is formulated for a Hausdorff space, in the homework please find where in the proof we use it, and generalize this theorem for non-Hausdorff spaces.

Local compactness is not preserved by continuous surjections. For example, if $(X, \mathscr{T})$ is any non-locally compact space, map $f : X \to X$, the identity map. Then f is a $(\mathscr{D}, \mathscr{T})$ continuous surjection, where $\mathscr{D}$ is the discrete topology on X. But, while $(X, \mathscr{D})$ is locally compact, $(X, \mathscr{T})$ is not by choice.
However, a continuous open map onto a Hausdorff space does preserve local compactness:

Theorem 52.3. *Let $(Y, \mathscr{M})$ be a Hausdorff space. Let $f : X \to Y$ be a $(\mathscr{T}, \mathscr{M})$ continuous, open surjection. Let X be $\mathscr{T}$-locally compact, then Y is $\mathscr{M}$-locally compact.*

Proof. For given $y \in Y$ choose $x \in X$ so that $f(x) = y$ and choose a relatively compact neighborhood $U(x)$. Because f is an open map, $f(U)$ is a neighborhood of y, and because $f(\overline{U})$ is compact (by Theorem (48.1)), we find from $\overline{f(U)} \subset \overline{f(\overline{U})} = f(\overline{U})$ that $\overline{f(U)}$ is compact. □

In the homework please explain why we need $(Y, \mathscr{M})$ be a Hausdorff space in Theorem (52.2).

Definition 52.2. Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then, A is $\mathscr{T}$-locally compact if and only if A is $\mathscr{T}_A$-locally compact.

Proposition 52.1. Let $(X, \mathscr{T})$ be a local compact space and $A \subset X$, then A is locally compact if and only if for any $x \in A$ there exists a $\mathscr{T}$-neighborhood V of x such that $A \cap (\overline{A \cap V})$ is $\mathscr{T}$-compact.

Proof is left as a homework.

Example 52.6. $[0, 1[$ *is a locally compact subset of E^1. To see this choose $x \in [0, 1[$ and consider two cases:*

1. *$x \neq 0$. Then choose $\delta > 0$ such that $]x - \delta, x + \delta[\subset]0, 1[$. Then, $]x - \delta, x + \delta[$ is a $\mathscr{T}$-neighborhood of x, and*

$$[0,1[\cap(\overline{[0,1[\cap]x-\delta,x+\delta[}) = [0,1[\cap([x-\delta,x+\delta]) \\ = [x-\delta,x+\delta] \textit{is compact in } E^1$$

2. *$x = 0$. Use $]-\frac{1}{2}, \frac{1}{2}[$ for V in Proposition (52.1). We have*

$$[0,1[\cap(\overline{[0,1[\cap]-\frac{1}{2},\frac{1}{2}[}) = [0,1[\cap\overline{[0,\frac{1}{2}[} = [0,1[\cap[0,\frac{1}{2}] = [0,\frac{1}{2}]$$

   and this is compact in E^1

Example 52.7. *A subspace of a locally compact space is not necessarily locally compact. For example, the set of irrationals $\mathscr{I}$ is not locally compact in E^1, although E^1 is locally compact.*
To see this, let V be any neighborhood of π. If $\mathscr{I} \cap (\overline{\mathscr{I} \cap V})$ is compact, then $\mathscr{I} \cap (\overline{\mathscr{I} \cap V})$ is bounded and closed. For some $\varepsilon > 0$, $]\pi - \varepsilon, \pi + \varepsilon[\subset V$. Choose any rational y with $\pi - \varepsilon < y < \pi + \varepsilon$. Then y is a cluster point of $\mathscr{I} \cap (\overline{\mathscr{I} \cap V})$, but $y \notin \mathscr{I} \cap (\overline{\mathscr{I} \cap V})$, as $y \notin \mathscr{I}$. This means that $\mathscr{I} \cap (\overline{\mathscr{I} \cap V})$ is not closed after all, a contradiction. Then $\mathscr{I} \cap (\overline{\mathscr{I} \cap V})$ cannot be compact, so $\mathscr{I}$ is not locally compact.
Of course, there is nothing special about π in this argument - any irrational will do.

As with subspaces, a product of locally compact spaces need not be locally compact. If, however, the coordinate spaces are Hausdorff, and if enough of them are compact, then the product will be locally compact.

Theorem 52.4. $\prod\{Y_\alpha \mid \alpha \in \mathscr{A}\}$ *is locally compact if and only if all the* Y_α *are locally compact Hausdorff spaces and at most finitely many are not compact.*

Proof. Assume the condition holds. Given $\{y_\alpha\} \in \prod_\alpha Y_\alpha$, for each of the at most finitely many indices for which Y_α is not compact, choose a relatively compact neighborhood $V_{\alpha_i}(y_{\alpha_i})$; then $\langle V_{\alpha_1}, \cdots, V_{\alpha_n}\rangle$ is a neighborhood of $\{y_\alpha\}$ and $\overline{\langle V_{\alpha_1}, \cdots, V_{\alpha_n}\rangle} = \langle \overline{V_{\alpha_1}}, \cdots, \overline{V_{\alpha_n}}\rangle$ is compact. Conversely, assume $\prod_\alpha Y_\alpha$ to be locally compact; since each projection p_α is a continuous open surjection, each Y_α is certainly locally compact. But also, choosing any relatively compact open $V \subset \prod_\alpha Y_\alpha$, each $p_\alpha(\overline{V})$ is compact, and since $p_\alpha(\overline{V}) = Y_\alpha$ for all but at most finitely many indices α, the result follows. $\square$

There is an important connection between local compactness and Alexandroff's Theorem. A one point compactification of a non-compact space is Hausdorff exactly when the space is Hausdorff and locally compact.

Theorem 52.5. *Let* $(X, \mathscr{T})$ *be a non-compact space and* $(Y, \mathscr{M})$ *be an Alexandroff one-point compactification of* $(X, \mathscr{T})$*. Then* $(Y, \mathscr{M})$ *is a Hausdorff space if and only of* $(X, \mathscr{T})$ *is Hausdorff and locally compact.*

Using Theorem (52.5) we can prove an important separation theorem:

Theorem 52.6. *Every locally compact Hausdorff space is completely regular.*

Proof. Let $(X, \mathscr{T})$ be a locally compact Hausdorff space and the space $(Y, \mathscr{M})$ be an Alexandroff one-point compactification of $(X, \mathscr{T})$. Since $(Y, \mathscr{M})$ is a compact Hausdorff space, then $\mathscr{M}$ is a normal topology (by Theorem (49.3)). By Urysohn's lemma $(Y, \mathscr{M})$ must be completely regular. Then $(X, \mathscr{T})$ is a subspace of a completely regular space so is also completely regular. $\square$

In the homework please explain why every subspace of completely regular space is also completely regular.

53 Countable compactness

The definition of compactness requires that each open cover be reducible to a finite cover. If this is relaxed to the reduction of just countable open covers, then we have a weaker condition called countable compactness.

Definition 53.1. $(X, \mathscr{T})$ is countably compact if and only if each countable open cover of X can be reduced to a finite open cover of X.

It is obvious that compactness implies countable compactness.

Theorem 53.1. *If $(X, \mathscr{T})$ is a compact space, then it countably compact.*

Countable compactness is characterized by the behavior of sequences only, rather than arbitrary filters, as the following equivalent formulations show:

Theorem 53.2. *The following three properties are equivalent:*

1. *Y is countably compact.*

2. ***(Bolzano-Weierstrass property)*** *Every countably infinite subset of Y has at least one cluster point.*

3. *Every sequence in Y has an accumulation point.*

Sometimes in the literature the third property is taken as a definition:

Definition 53.2. $(X, \mathscr{T})$ is Bolzano-Weierstrass compact if and only if each sequence in X has a limit point in X.

In the next lectures we shall return to the notion of countable compactness, in particular we will prove that compactness and countable compactness are equivalent in metric spaces.

54 Sequential compactness

A space is sequentially compact if each sequence has a convergent subsequence. For example, let S be a sequence in E^1, whose elements are $S_n = 1/n$ for $n = 1, 2, \ldots$. Let $V_n = 1/2n$ for $n = 1, 2, \ldots$. Then V is also a sequence in E^1 but may be considered as a subsequence of S in the sense that $V_n = S_{2n}$. That is, $V = S \circ \varphi$, where φ is an increasing function on Z^+ to Z^+. this motivates the following definition:

Definition 54.1. Let S be a sequence in X. Then, V is a subsequence of S if and only if for some $\varphi : Z^+ \to Z^+$, we have: $\varphi(n) < \varphi(m)$ if $n < m$, and $V = S \circ \varphi$.

Definition 54.2. $(X, \mathscr{T})$ is sequentially compact if and only if each sequence in X has a convergent subsequence.

It is obvious that

Theorem 54.1. *If $(X, \mathscr{T})$ is sequentially compact then X is countably compact.*

However, it is not true that countable compactness implies sequential compactness. But, as we will see in the next section in some spaces countable compactness does imply sequential compactness.

55 1°-countable spaces

In 1° countable spaces, sequences and subsequences not only behave properly, but also adequate to express all topological concepts; this accounts for their great utility in the metric spaces and elementary analysis.

Definition 55.1. A topological space Y is 1° countable (or satisfies the first axiom of countability) if with each $y \in Y$ there is given an at most countable family $\{U_n(y) \mid n \in Z^+\}$ of neighborhoods with the property: For each open $G \supset y$, there is some $U_n(y) \subset G$. (expressed briefly: if Y has a countable basis at each point)

Proposition 55.1. Let $\varphi : Z^+ \to X$ be a sequence in a Hausdorff space. Then:

1. $\varphi \to y_0$ if and only if each subsequence φ' of φ contains a subsequence φ'' such that $\varphi'' \to y_0$.
2. Let X be 1° countable, then φ accumulates at x_0 if and only if there is some subsequence $\varphi' \to x_0$.

Proof.

1. The "only if" is trivial. "If": Assume that $\varphi \nrightarrow x_0$; then $\exists U(x_0)\forall T_n : \varphi(T_n) \nsubseteq U$. Proceeding by induction, let $n_1 \geqslant 1$ be the first integer in T_1 such that $\varphi(n_1) \notin U$, and assuming that $n_1 < \ldots < n_k$ have been obtained, let n_{k+1} be the first integer in T_{n_k+1} such that $\varphi(n_{k+1}) \notin U$. Let φ' be the subsequence defined by $\varphi'(k) = \varphi(n_k)$. Since $\varphi'(T_n) \subset \mathbf{C}U$ for every n, no subsequence of φ' can converge to y_0.
2. Only the existence of φ' requires proof. Let $U_1 \supset U_2 \supset \cdots$ be a countable basis at x_0 and assume that φ accumulates at x_0; then $\forall U_i\ \forall T_m :\ \varphi(T_m) \cap U_i \neq 0$. Proceeding by induction, let $n_1 \geqslant 1$ be the first integer in T_1 with $\varphi(n_1) \in U_1$, and assuming that $n_1 < \cdots < n_k$ have been defined, let n_{k+1} be the first integer in T_{n_k+1} with $\varphi(n_{k+1} \in U_{k+1}$. The subsequence $\varphi'(k) = \varphi(n_k)$ evidently converges to x_0.

□

Remarks: Remind definitions from calculus: Let Z^+ be the set of positive integers. A sequence in a space Y is a map $\varphi : Z^+ \to Y$. We say

1. φ converges to y_0 (written $\varphi \to y_0$) if:
$$\forall U(y_0)\exists N\forall n \geqslant N : \varphi(n) \in U$$
2. φ accumulates at y_0 if:
$$\forall U(y_0)\forall N\exists n \geqslant N : \varphi(n) \in U$$

Theorem 55.1. *Let $(X, \mathscr{T})$ be 1° countable and let $A \subset X$. Then $x \in \overline{A}$ if and only if there is a sequence on A converging to x.*

Then for maps of 1° countable spaces we have:

Proposition 55.2. Let $(X, \mathscr{T})$ be 1° countable Hausdorff space. Then for Y arbitrary, and $f : X \to Y$:

1. $f(\mathscr{U}(x))$ (where $\mathscr{U}(x)$ if filter converges to y_0) converges to y_0 if and only if $f(x_n \to y_0$ for each sequence $x_n \to x$.

2. f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for each sequence $x_n \to x_0$.

3. Let Y be regular, $D \subset X$ dense, and $g : D \to Y$ continuous. Then g is extendable to a continuous $G : x \to Y$ is and only if each $x \in X$ and all sequences $\{d_n\} \subset D$ converging to x, the sequences $\{f(d_n)\}$ all converge and to the same limit.

Sequential and countable compactness are equivalent in 1° countable spaces:

Theorem 55.2. *Let $(X, \mathscr{T})$ be 1° countable. Then X is sequentially compact if and only if X is countable compact.*

So far we have learned the following relations between different kinds of compactness:

- Compactness implies countable compactness
- Sequential compactness implies countable compactness
- Countable compactness implies sequential compactness in 1° countable spaces.

Finally, all the topological these topological concepts are invariants:

Theorem 55.3. *Let $(X, \mathscr{T}) \cong (Y, \mathscr{M})$, then*

1. *X is countable compact $\Leftrightarrow$ Y is countable compact.*

2. *X is sequentially compact $\Leftrightarrow$ Y is sequentially compact.*

3. *X is 1° countable $\Leftrightarrow$ Y is 1° countable.*

56 2°-countable, separable and Lindelöf spaces

We shall briefly sketch three additional concepts which play an important role in set topology.

Definition 56.1. $(X, \mathscr{T})$ is 2°-countable if and only if $\mathscr{T}$ has a countable basis.

Theorem 56.1. *If $(X, \mathscr{T})$ is 2°-countable, then $(X, \mathscr{T})$ is 1°-countable.*

Proof is left as a homework.

We will see later that every metric space is 1°-countable but nit necessary 2°-countable. In fact, a metric space is 2°-countable exactly when it has a countable subset closure is the whole space.

Definition 56.2. A is dense in $(X, \mathscr{T})$ if and only if $\overline{A} = X$.

Definition 56.3. $(X, \mathscr{T})$ is separable if and only if there is a countable dense subset of X.

If A is dense in X, then each element of X can be approximated arbitrary closely (in sense of the topology) by elements of A. For example, the rational are dense in E^1, and the polynomials are dense in the sup-norm space $C([a, b])$. This is the Weierstrass Approximation Theorem. A non-compact space is also dense in its one point compactification.

Theorem 56.2. *Let $(X, \mathscr{T})$ be 2°-countable, then $(X, \mathscr{T})$ is separable.*

But it is not true in general that separable implies 2°-countable.
The notion of a Lindelöf space is in a sense complementary to that of countable compactness. A space is Lindelöf if each open cover has a countable reduction.

Definition 56.4. $(X, \mathscr{T})$ is Lindelöf if and only if each open cover can be reduced to a countable open cover.

Theorem 56.3.

1. *If $(X, \mathscr{T})$ compact, then $(X, \mathscr{T})$ is Lindelöf.*
2. *If $(X, \mathscr{T})$ is Lindelöf and countable compact, then $(X, \mathscr{T})$ is compact.*

Example 56.1. *Give the set R of real numbers the topology $\mathscr{T}$ generated by the half-open intervals $]a, b]$. Then, $\{(x, y) \mid x \text{ and } y \text{ are rational}\}$ is a countable, dense subset, so that $R \times R$ is separable in the product topology. However, $R \times R$ is not Lindelöf.*
To see this, suppose $R \times R$ is Lindelöf. It is easy to check each closed subspace would then also be Lindelöf (but check it!). Now, $\{(r, -r) \mid r \text{ is irrational}\}$ is a closed subspace of $R \times R$, so would be Lindelöf. But, for any rational number r, $]r - 1, r] \times] - r - 1, -r] \cap \{(y, -y) \mid y \text{ is irrational}\} = \{(r, -r)\}$ is a neighborhood of $(r, -r)$, so that the subspace $\{(r, -r) \mid r \text{ is irrational}\}$ is discrete. And obviously an uncountable, discrete space cannot be Lindelöf (take an open cover of singletons).

It is also possible for a space to be Lindelöf and neither compact, countable compact, nor 2°-countable.

Example 56.2. *Let R be the set of real numbers, and $\mathscr{T}$ the topology generated by the sets $]-\infty, x]$. The set $(R, \mathscr{T}$ is Lindelöf, but neither compact, countable compact, nor 2°-countable.*

Theorem 56.4. *Every 2°-countable space is Lindelöf.*

Theorem 56.5. *Regular Lindelöf space is normal.*