Ma 635. Real Analysis I. HW problems

HW 1 (due 09/7):

1. (Kolm, p.8 #2) Show that in general $(A - B) \cup B \neq A$.

Solution: Consider $A = \{1, 2\}$, $B = \{2, 3\}$. Then $A - B = \{1\}$ and $(A - B) \cup B = \{1, 2, 3\} \neq A$.

2. (Kolm, p.9 #6) Let A_n be the set of all positive integers divisible by n. Find the sets

$$\text{(a)} \quad \bigcup_{n=2}^{\infty} A_n; \qquad \text{(b)} \quad \bigcap_{n=2}^{\infty} A_n.$$

Solution. (a) In view of $n \in A_n$ we obtain $\bigcup\limits_{n=2}^{\infty} A_n = \{2, 3, 4, \ldots\} = \mathbf{N} - \{1\}$.

(b) Let $k \in \bigcap\limits_{n=2}^{\infty} A_n$ and $p > k$ be a prime number. Then $k \notin A_p$ and $k \notin \bigcap\limits_{n=2}^{\infty} A_n$. Finally, $\bigcap\limits_{n=2}^{\infty} A_n = \emptyset$.

3. (Kolm, p.9 # 8) Let A_α be the set of points lying on the curv $y = 1/x^\alpha$, $0 < x < \infty$. What is $\bigcap\limits_{\alpha \geq 1} A_\alpha$?

Hint: plot the graphs of the functions for different α. Answer: just one point $(1, 1)$.

4. (Kolm, p. 19 # 2). Let M be any infinite set and A any countable set. Prove that $M \sim M \cup A$.

Solution: If M is countable then clearly $M \sim M \cup A$ since both sets are countable. Let's consider $|M| > \aleph_0$. Let $M_A \subset M$ be a countable subset of M. Then we establish a bijection (1-1 and onto correspondence) between M_A and $M_A \cup A$ and the natural correspondence between $M \setminus M_A$ and $(M \cup A) \setminus (M_A \cup A)$. Thus, we obtain a bijection between M and $M \cup A$.

5. (Kolm, p. 19 # 6) Prove that the set F of all real functions defined on a set M has a greater cardinality than $|M|$.

Solution. The set C of all characteristic functions (i.e., taking only the values 0 and 1) on M is equivalent to $P(M)$ $-$ the set of all subsets of M. Since $C \subset F$ then $|F| \geq |C| = |P(M)| = 2^{|M|} > |M|$.