Ma 635. Real Analysis I. Hw2 (due 09/14). Solutions.

1. [2] p. 45 # 1
Given a metric space (X, d), prove that
(a) $|d(x, z) - d(y, u)| \le d(x, y) + d(z, u)$
(b) $|d(x, z) - d(y, z)| \le d(x, y)$

2. [2] p. 45 # 5
Prove that the metric in $(-\infty, +\infty)$, $d_\infty(x, y) = \max\limits_{1\le k\le n} |x_k - y_k|$ is the limiting case of the metric $d_p(x, y) = \left(\sum\limits_{k=1}^{n} |x_k - y_k|^p\right)^{1/p}$ as $p \to \infty$.

3. [2] p. 45 # 8
Exhibit an isometry between the spaces $C[0, 1]$ and $C[1, 2]$.

4. [2] p. 54 # 3
Prove that if $x_n \to x$, $y_n \to y$ as $n \to \infty$ then $d(x_n, y_n) \to d(x, y)$.

5. [2] p. 54 # 7
Show that 1/4 belongs to the Cantor set.

6. [2] p. 65 # 2
Prove that space $m = l_\infty$ of bounded sequences with metric $d(x, y) = \sup\limits_{1\le k\le\infty} |x_k - y_k|$ is complete.

7. [2] p. 65 # 4
Suppose metric space R is complete, and let $\{A_n\}$ be a sequence of closed subsets of R nested in the sense that

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

Let also the diameters tend to zero: $\lim\limits_{n\to\infty} d(A_n) = 0$. Prove that the intersection $\bigcap\limits_{n=1}^{\infty} A_n$ is nonempty.

8. [1] p. 38 # 1
Show that

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

defines a metric on $(0, \infty)$.

9. [1] p. 38 # 6
If d is any metric on M, show that $\rho(x, y) = \sqrt{d(x, y)}$, $\sigma(x, y) = \frac{d(x,y}{1+d(x,y)}$, and $\tau(x, y) = \min\{d(x, y), 1\}$ are also metrics on M.

10. [1] p. 39 # 11
Let R^∞ be the space of all infinite dimensional vectors $\{x_n\}_{n=1}^{\infty}$. Show that the expression

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on R^∞.

11. [1] p. 39 # 12
Check that $d(x, y) = \sup\limits_{a\le t\le b} |x(t) - y(t)|$ defines a metric on $C[a, b]$, the space of all continuous

functions defined on the closed interval $[a, b]$.

12. [1] p. 42 # 23
The subset of l_∞ consisting of all sequences that converge to 0 is denoted by c_0. Show that we have the following proper set inclusions: $l_1 \subset l_2 \subset c_0 \subset l_\infty$.

13. [1] p. 46 # 34
If $x_n \to x$ in (M, d), show that $\forall y \in M,\ d(x_n, y) \to d(x, y)$.

14. [1] p. 46 # 37
A Cauchy sequence with a convergent subsequence converges.

(bonus 1) [2] p. 53 #1
Give an example of a metric space R and two open balls $B_{r_1}(x)$ and $B_{r_2}(x)$ in R such that $B_{r_1}(x) \subset B_{r_2}(y)$ although $r_1 > r_2$.

(bonus 2) [2] p. 65 # 6
Give an example of a complete metric space R and a nested sequence $\{A_n\}$ of closed subsets of R such that

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Reconcile this example with Problem 4.

References

[1] Carothers N.L., *Real Analysis.* Cambridge University Press, 2000.
ISBN 0521497493 or ISBN 0521497566.

[2] Kolmogorov, A.N., and Fomin, S.V., *Introductory Real Analysis.* Dover, 1970.
ISBN 0486612260.

[3] Haaser, N.B., and Sullivan, J.A., *Real Analysis.* Dover, 1991.
ISBN 0486665097.

[4] Rudin, W., *Real and Complex Analysis*, 3d ed. McGraw-Hills, 1987.

[5] Folland, G.B., *Real Analysis.* Wiley, 1984.

[6] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. 1. Functional Analysis.* Academic Press 1972.

[7] Oxtoby, J.C., *Measure and Category. A survey of the Analogies between Topological and Measure Spaces.* Springer-Verlag, 1971.