Ma 635. Real Analysis I. Hw3

HW 3 (due 09/21):
1. [1] p. 45 # 27
Show that $\mathrm{diam}(B_r(x)) \le 2r$ and give an example where strict inequality occurs.
Solution. By definition, $B_r(x) = \{y : \ d(x, y) < r\}$. Then

$$\mathrm{diam}(B_r(x)) = \sup_{a,b \in B_r(x)} d(a, b) \le \sup_{a,b \in B_r(x)} d(a, x) + d(b, x) \le 2r.$$

The strict inequality holds, say, for the metric space of three points $\{x, y, z\}$ with $d(x, y) = 2$, $d(x, z) = 3$, $d(y, z) = 4$. Then $\mathrm{diam}(B_{2.1}(x)) = 2 < 2.1 = r < 2r$.

2. [1] p. 45 # 29
Prove that A is bounded iff $\mathrm{diam}(A) < \infty$.
Solution. By definition, a set is bounded if it is contained in a ball. From the previous problem, the diameter is less than double radius. Conversely, if $\mathrm{diam}(A) < \infty$ then $sup_{a,b\in A} d(a, b) < \infty$. Let us fix $a \in A$. Then $A \in B_{\mathrm{diam}(A)}(a)$, and, hence, A is bounded.

3. [1] p. 45 # 31
Give an example where $\mathrm{diam}(A \cup B) >$diam$(A)+$diam$(B)$.
Solution. Let $A = (0, 1)$, $B = (2, 3)$.

4. [1] p. 55 # 3
Two metrics are equivalent if they generate the same convergent sequences; that is, $d_1(x_n, x) \to 0$ if and only if $d_2(x_n, x) \to 0$. Prove that equivalent metrics generate the same open sets.
Solution. Equivalent metrics generate the same closed sets. Really, closed sets contain all limit points. Therefore if a set is closed in one metric, then it is also closed in another metric.

5. [1] p. 55 # 11
Let $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where the kth entry is 1 and the rest are 0s. Show that $\{e^{(k)} : \ k \ge 1\}$ is closed in l_1.
Solution. If a set contains all limit points then it is closed. The only type of Cauchy sequences in the given set are stationary sequences like $\{e^{(k)}, e^{(k)}, e^{(k)}, \ldots, e^{(k)}, \ldots\}$. Their limit is just $e^{(k)}$ that belongs to the given set.

6. [1] p. 55 # 14
Show that the set $A = \{x \in l_2 : \ |x_n| \le 1/n, \ n = 1, 2, \ldots\}$ is a closed set in l_2 but that $B = \{x \in l_2 : \ |x_n| < 1/n, \ n = 1, 2, \ldots\}$ is not an open set.
Solution. There is no open ball $B_\varepsilon(y)$ inside B. Really, $x = y + (0, 0, \ldots, 0, \varepsilon/2, 0, \ldots) \in B_\varepsilon(y)$ but $x \notin B$ if the entry for ε is sufficiently far from the "beginning".

7. [1] p. 57 # 17
Show that A is open if and only if (iff) $A^o = A$ and that A is closed iff $\overline{A} = A$.
Solution. By definition, the internal part A, A^o, is the biggest open subset of A. If it coincides with A then A is open. If A is open then A is the biggest subset of itself, and then $A^o = A$.
By definition, the closure $\overline{A}$ is the set of all contact points of A (or, the same, the union of A and all its limit points). Consequently, $\overline{A}$ contains all its limit points, which implies that $\overline{A}$ is closed. Vice versa, if A is closed then it contains all limit points and is closed.

8. [1] p. 57 # 22
True or false? $(A \cup B)^o = A^o \cup B^o$?
Solution. False. Let $A = [0, 1]$, $B = [1, 2]$. But $(0, 2) \ne (0, 1) \cup (1, 2)$.

9. [1] p. 57 # 26
Prove that $d(x, A) = 0 \Leftrightarrow x \in \overline{A}$.
Solution. ($\Rightarrow$) By definition, $d(x, A) = \inf\limits_{a\in A} d(x, a)$. Therefore $\forall \varepsilon > 0 \ \exists a \in A$ such that $d(x, a) < d(x, A) + \varepsilon = \varepsilon$. For every $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ we pick up a sequence $\{a_1, a_2, a_3, \ldots\} \subset A$, which converges to x. Then x is a limit

point of A and $x \in \overline{A}$.
($\Leftarrow$) If $x \in \overline{A}$ then $\exists\{a_1, a_2, \ldots\} \subset A$, $a_n \to x$. That is, $d(a_n, x) \to 0$ and, hence, $\inf\limits_{a\in A} d(x,a) = 0$.

10. [1] p. 58 # 33
Obvious. Consider a sequence $\varepsilon_n \to 0$.

11. [1] p. 59 # 43
Show that the boundary $\partial A = \overline{A} \setminus A^o$ is closed.
Solution. If ∂A isn't closed then $\exists$ a Cauchy sequence $\{x_n\} \subset \partial A$ with no limit x in A. Since $\overline{A}$ is closed, then $x \in \overline{A}$. However, since $x \notin \partial A$ then $x \in A^o$. Every ball centered at x contains a point from $\{x_n\} \not\subset A^o$. Consequently, A^o is not open. This contradiction proves that the assumption was wrong and ∂A is actually closed.

12. [1] p. 59 # 46
solved in Kolmogorov book.

13. [1] p. 59 # 48
solved in Kolmogorov book.

14. [1] p. 59 # 50
l_∞ is not separable.
Solution. The sequences of zeros and ones form an uncountable subset in l_∞. The distance between any two points from this subset is equal to 2. Let us consider non-intersecting balls centered at the elements of this subset with radius 1/2. If there were a dense countable subset in l_∞ then each of the balls had to contain a point from that dense subset. Since the number of balls is uncountable then that dense subset cannot be countable. Consequently, l_∞ is not separable.
15. [1] p. 59 # 54
16. [1] p. 59 # 58
Solution. The sum of lengths of intervals I_n is equal to 1. Therefore U cannot cover $(-\infty, +\infty)$ and, hence, is a proper subset. U is open as a union of open sets. U is dense since contains all rational points, which are dense. U^c is nowhere dense, otherwise $\exists B_\varepsilon(x) \subset \overline{U^c}$. But any interval B_ε contains a rational point and, thus, cannot be a subset of $\overline{U^c}$. Consequently, U^c is nowhere dense.

17. [1] p. 60 # 60
solved in Kolmogorov book

18. [1] p. 64 # 1(iii-vi)
19. [1] p. 65 # 5
Solution. χ_A is also continuous in any point from $\text{int}A^c$.
χ_A is discontinuous at ∂A. The only continuous characteristic function is χ_R.

20. [1] p. 65 # 13

References

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