Ma 635. Real Analysis I. Hw4. Solutions.

HW 4 (due 09/28):

1. [1] p. 66 # 25

A function $f: \ (M, d) \mapsto (N, \rho)$ is called Lipshitz if $\exists K, \ \forall x, y \in M, \ \rho(f(x), f(y)) \leq K d(x, y)$. Prove that Lipshitz mapping is continuous.

Solution. $\rho(f(x_n, f(x)) \leq K d(x_n, x) \to 0$ if $x_n \to x$ as $n \to \infty$.

2. [1] p. 66 # 28

Let $g: l_2 \mapsto R$, $g(x) = \sum\limits_{n=1}^{\infty} \frac{x_n}{n}$. Is g continuous?

Solution. Yes.

$$\begin{aligned} \left| g\left(x^{(k)}\right) - g(x) \right| &\leq \sum_{n=1}^{\infty} \frac{\left| x_n^{(k)} - x_n \right|}{n} \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=1}^{\infty} \left| x_n^{(k)} - x_n \right|^2 \right)^{1/2} = \frac{\pi}{\sqrt{6}} \left\| x^{(k)} - x \right\|_{l_2} \to 0 \end{aligned}$$

as $k \to \infty$. The second inequality is just Schwartz inequality.

3. [1] p. 66 # 29

Fix $y \in l_\infty$ and define $h: \ l_1 \mapsto l_1$ by $h(x) = (x_n y_n)_{n=1}^{\infty}$. Show that h is continuous.

Solution.

$$d_1(h(x), h(z)) = \sum_{n=1}^{\infty} |h(x)_n - h(z)_n| = \sum_{n=1}^{\infty} |x_n y_n - z_n y_n| = \sum_{n=1}^{\infty} |x_n - z_n| \cdot |y_n| \leq \|y\|_{l_\infty} \|x - z\|_{l_1} \to 0$$

as $x \to z$ in l_1.

4. [1] p. 71 # 43

Prove that two metrics d and ρ on a set M are equivalent iff the identity map on M is a homeomorphism from (M, d) to (M, ρ).

Solution. ($\Longrightarrow$) By definition, if two metrics are equivalent then $d(x_n, x) \to 0 \Longleftrightarrow \rho(x_n, x) \to 0$. We consider the identity map $I: \ M \mapsto M$, $I(x) = x$. It is continuous, because $\rho(I(x_n), I(x)) = \rho(x_n, x) \to 0$ if $d(x_n, x) \to 0$.

($\Longleftarrow$) If I is a homeomorphism then both I and I^{-1} are continuous. Then $\rho(I(x_n), I(x)) = \rho(x_n, x) \to 0$ if $d(x_n, x) \to 0$. Similarly, we can show the inverse.

5. [1] p. 71 # 45

Prove that $\mathbb{N}$ with usual metric is homeomorphic to $M = \{\frac{1}{n}\}_{n=1}^{\infty}$ with usual metric.

Solution. Consider $f: \ \mathbb{N} \mapsto M$, $f(n) = \frac{1}{n}$. Obviously, $\exists f^{-1}$ and both f and f^{-1} are continuous. In fact, if a sequence from $\mathbb{N}$ converges then it is stationary. Therefore its image is also convergent to the image of the limit, and f is continuous.

6. [1] p. 71 # 49

Let V be a normed vector space. Given a fixed vector $y \in V$, show that the map $f(x) = x + y$ (translation ny y) is an isometry on V. Given a nonzero scalar $\alpha \in R$, show that the map $g(x) = \alpha x$ (dilation by α) is a homeomorphism on V.

Solution. $\|f(x) - f(z)\| = \|(x + y) - (z + y)\| = \|x - z\|$. So, f is isometry.

$\|g(x_n) - g(x)\| = \|\alpha x_n - \alpha x\| = |\alpha| \cdot \|x_n - x\|$. Therefore, $\|g(x_n) - g(x)\| \to 0$ if $\|x_n - x\| \to 0$, and g is continuous. The same way shows the continuity of g^{-1}.

7. [1] prove theorem 5.10 (p. 75)

8. [1] p. 75 # 62

9. [1] p. 94 # 13
Show that R endowed with the metric $\rho(x,y) = |\arctan x - \arctan y|$ is not complete. How about $\tau(x,y) = |x^3 - y^3|$?

10. [1] p. 94 # 14
If we define $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m,n \in N$, show that d is equivalent to the usual metric on N but (N,d) is not complete.

11. [1] p. 94 # 19
Show that c_0 is closed in l_∞.
Solution. Let $\{x^{(n)}\}_{n=1}^\infty \subset c_0$ is a converging (in l_∞) sequence to x: $\lim_{n\to\infty} \|x^{(n)} - x\|_\infty = 0$. By the definition of limit, this means that

$$\forall \varepsilon > 0 \; \exists N \; \forall n > N : \sup_{1 \le k \le \infty} |x_k^{(n)} - x_k| < \varepsilon. \tag{1}$$

From (1) we obtain the coordinate-wise convergence $x^{(n)} \to x$, $n \to \infty$. Since $\forall\, n,\ \ x^{(n)} \in c_0$, then

$$\forall\, n \; \exists K = K(n) \; \forall k > K, \; |x_k^{(n)}| < \varepsilon. \tag{2}$$

In view of (1), for an arbitrary $\varepsilon > 0$ we pick up N such that for all $n > N$ and all k, $|x_k^{(n)} - x_k| < \varepsilon/2$. After that, we fix any $n_0 > N$ and find K (depending on n_0 and ε) such that from (2) $\forall k > K$, $|x_k^{(n_0)}| < \varepsilon/2$.

Finally, $|x_k| \le |x_k^{(n_0)} - x_k| + |x_k^{(n_0)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ subject to $k > K$. This implies $x_k \to 0$ as $k \to \infty$ and, hence, $x \in c_0$. Then c_0 is closed in l_∞.

12. [1] p. 94 # 20

13. [1] p. 94 # 21

14. [1] p. 102 # 42
Define $T:\ C[0,1] \mapsto C[0,1]$ by $Tf(x) = \int_0^x f(t)dt$. Show that T^2 is a contraction. What is its fixed point?

15. To solve equation $f(x) = \lambda \int_0^2 (x+y)f(y)dy$, we consider $f \in C[0,2]$ and $F(f)(x) = 1 + \lambda \int_0^2 (x+y)f(y)dy$. Find λ at which $F:\ C[0,2] \mapsto C[0,2]$ is contractive. For initial value $f(x) = 0$ perform 3 iterations to approach the solution $f(x)$.

(!) Please read the corresponding sections of the textbook.

References

[1] Carothers N.L., *Real Analysis.* Cambridge University Press, 2000.
ISBN 0521497493 or ISBN 0521497566.

[2] Kolmogorov, A.N., and Fomin, S.V., *Introductory Real Analysis.* Dover, 1970.
ISBN 0486612260.

[3] Haaser, N.B., and Sullivan, J.A., *Real Analysis.* Dover, 1991.
ISBN 0486665097.

[4] Rudin, W., *Real and Complex Analysis*, 3d ed. McGraw-Hills, 1987.

[5] Folland, G.B., *Real Analysis.* Wiley, 1984.

[6] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. 1. Functional Analysis.* Academic Press 1972.

[7] Oxtoby, J.C., *Measure and Category. A survey of the Analogies between Topological and Measure Spaces.* Springer-Verlag, 1971.