Ma 635. Real Analysis I. Lecture Notes 1.

I. ELEMENTS of DISCRETE MATHEMATICS

1.1 A mapping $f: A \mapsto B$ is a *function* if $\forall x \in A \exists! y \in B$ such that $f(x) = y$. A is the *domain* of f, B is its *codomain.* The values of $y \in B$ that have at least one pre-image (inverse image) $x \in A$ form the *range* of f, $R(f) \subset B$.

1.2 Function $f(x)$ is *one-to-one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

1.3 Show that f is 1-1 if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

1.4 Function $f(x)$ is *onto* if its range covers the entire codomain: $R(f) = B$. Function f is *bijection* if it is both 1-1 and onto.

1.5 An *equivalence relation* on set X is a relation, which is reflexive ($x \sim x$), symmetric ($x \sim y \Leftrightarrow y \sim x$), and transitive ($x \sim y,\ y \sim z \Rightarrow x \sim z$).

1.6 Equivalence relation splits X into a collection of non-intersecting subsets (partition of X). These subsets are called *equivalence classes.* Each equivalence class $[x]$ can be characterized by its representative x. Equivalence classes form *quotient space* $X/\sim$.

1.7 $p \to q \Longleftrightarrow \sim q \to \sim p$ (implication is equivalent to its contrapositive).

II. ELEMENTS of SET THEORY

2.1 Symmetric difference of sets $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

2.2 Sets A and B are *equivalent* ($A \sim B$) if $\exists$ a bijection $f: A \mapsto B$. If $A \sim B$ then A and B have the same cardinal number. All sets are partitioned into non-intersecting equivalence classes. To each of the equivalence class a *cardinal number* card(A) (or $|A|$) is assigned.

2.3 $|A| > |B|$ if $\exists$ bijection $f: A \mapsto B'$, $B' \subset B$ and $A \not\sim B$.

2.4 $|\mathbf{N}| = \aleph_0$, $|(0,1)| = \mathbf{c} > \aleph_0$.

2.5 $|\mathbf{Q}| = \aleph_0$.
Hint: consider the rational numbers as the points on the plane with integer coordinates and choose a trial that covers all the points.

2.6 $\mathbf{c} > \aleph_0$.
Hint: prove by contradiction using the diagonal process.

2.7 $|\mathbf{R}^n| = \mathbf{c}$.

2.8 If $|A| = \aleph_0$ then $|\underbrace{A \times A \times \cdots}_{\aleph_0 \text{ times}}| = \mathbf{c}$. If $|A| = \mathbf{c}$ then also $|\underbrace{A \times A \times \cdots}_{\aleph_0 \text{ times}}| = \mathbf{c}$.

2.9 $P(A)$ - *power set* (the set of all subsets of A), $|P(A)| = 2^{|A|}$.

2.10 $2^{|A|} > |A|$.
Hint: assume $2^{|A|} = |A|$ and consider the set X formed by the elements of A which do not belong to their "associated subsets".

2.11 $2^{\aleph_0} = \mathbf{c}$.

2.12 card(set of all real functions over A)$\geq 2^{|A|}$.
Particularly, card(set of all real functions on $\mathbf{R}$)$\geq 2^{\mathbf{c}}$.
Hint: consider the set of all *characteristic* functions that take the values 0 and 1 only

2.13 card(set of all real *continuous* functions on $\mathbf{R}$)$= \mathbf{c}$.
Hint: continuous functions are defined by their values on rational numbers whose cardinal number is $\aleph_0$. Then use problem 2.8.

III. ELEMENTS of CALCULUS

3.1 $A = \sup\limits_{x\in X} f(x)$ if
(a) A is an upper bound: $\forall\, x,\ f(x) \leq A$;
(b) this upper bound is **exact**: $\forall\, \varepsilon > 0\ \exists\, x:\ f(x) > A - \varepsilon$.

3.2 $B = \inf_{x\in X} f(x)$ if (1) $\forall\, x,\ f(x) \geq A$ and (2) $\forall\, \varepsilon > 0\ \exists\, x:\ f(x) < A + \varepsilon$.

3.3 $A = \lim\limits_{n\to\infty} x_n$ if $\forall\varepsilon > 0\ \exists N = N(\varepsilon)\ \forall n > N:\ |x_n - A| < \varepsilon$ (any ε-neighborhood of A contains a "tail" of the sequence).

3.4 $A \neq \lim\limits_{n\to\infty} x_n$ if $\exists\, \varepsilon > 0\ \forall N\ \exists\, n > N:\ |x_n - A| \geq \varepsilon$.

3.5 $\{x_n\}$ is a *Cauchy sequence* if $\forall\varepsilon > 0\ \exists N\ \forall n, m > N:\ |x_n - x_m| < \varepsilon$. Any Cauchy sequence has a limit.

3.6 Any convergent sequence is a Cauchy sequence.
Hint: $|x_n - x_m| \leq |x_n - x| + |x_m - x| \to 0,\ n, m \to \infty$.

3.7 If a sequence is convergent then the limit is unique.
Hint: assume that there are two distinct limits.

3.8 $A = \lim\limits_{x\to x_0} f(x)$ if $\forall\varepsilon > 0\ \exists\delta\ \forall x,\ |x - x_0| < \delta:\ |f(x) - A| < \varepsilon$ (pre-image of any ε-neighborhood of A contains a δ-neighborhood of x_0).

3.9 $A = \lim\limits_{x\to x_0} f(x) \Leftrightarrow [x_n \to x_0] \longrightarrow [f(x_n) \to A]$.

3.10 $f(x)$ is *continuous* at $x = x_0$ if $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

3.11 $f(x)$ is *uniformly continuous* over $[a, b]$ if $\forall\varepsilon > 0\, \exists\delta\ \forall x',\ x'' \in [a, b]$:
$|x' - x''| < \delta \to |f(x') - f(x'')| < \varepsilon$.

3.12 Sequence of functions $f_n(x)$ is *pointwise convergent* to $f(x)$, $f_n \longrightarrow f$, if $\forall\, \varepsilon > 0\, \forall x\, \exists\, N\, \forall n > N\, |f_n(x) - f(x)| < \varepsilon$. Here N depends on x.

3.13 Sequence of functions $f_n(x)$ is *uniformly convergent* to $f(x)$, $f_n \rightrightarrows f$, if $\forall\, \varepsilon > 0\, \exists\, N\, \forall n > N\, \forall x\ |f_n(x) - f(x)| < \varepsilon$. Here N is independent of x.

3.14 Sequence of functions $f_n(x) = x^n$, $x \in [0, 1]$, is pointwise convergent to $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$ but is not uniformly convergent to $f(x)$.

3.15 $f_n \rightrightarrows f \iff \sup_x |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

3.16 Sequence of infinite-dimensional vectors (sequences) $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$ is *coordinatewise* convergent to $x = (x_1, x_2, \ldots)$ as $n \to \infty$ if $\forall \varepsilon > 0 \forall k \, \exists N = N(\varepsilon, k) \, \forall n > N : \; |x_k^{(n)} - x_k| < \varepsilon$.

3.17 Sequence of infinite-dimensional vectors $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$ is *uniformly* convergent to $x = (x_1, x_2, \ldots)$ as $n \to \infty$, $x^{(n)} \rightrightarrows x$, if $\forall \varepsilon > 0 \exists N = N \, \forall n > N \, \forall k \; : \; |x_k^{(n)} - x_k| < \varepsilon$.

3.18 Sequence $x^{(n)} = (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots)$ is coordinatewise convergent to $x = (1, 1, 1, \ldots)$ but it does not converge to x uniformly.

3.19 $x^{(n)} \rightrightarrows x \iff \sup_{1 \le k < \infty} |x_k^{(n)} - x_k| \to 0$ as $n \to \infty$.

Reading:
[1] sections 1, 2, 10;
[2] sections 1.1-1.4, 2.1-2.6, 3.1-3.8;
[3] sections 1.1-1.6, 2.3-2.5, 4.1, 4.5.

References

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