Universal Groups of Prees

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What is a pree?

The canonical example is an amalgam of groups with partial multiplication: $aa' = a'', bb' = b''$.

The partial multiplication has an identity, inverses, and is associative where possible.

This pree embeds in its universal group

$$\langle A \cup B \mid aa' = a'', bb' = b'' \rangle.$$  

Its Cayley diagram is essentially the link of the presentation complex.
A pree is a set $P$ with a partial binary operation such that

1. There is an identity, i.e., an element $1 \in P$ such that for all $a \in P$, the products $1a$ and $a1$ are defined and are both equal to $a$.

2. Each $a \in P$ has a 2-sided inverse.

3. If $ab$ and $bc$ are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case $(ab)c = a(bc)$.

It is not hard to show that inverses are unique in a pree. Further if $ab = c$, then $b = a^{-1}c$ etc.

Every group is a pree.

Every finitely presented group is the universal group of a finite pree.
Triangles

Each product $ab = c$ defined may be represented by a triangle.

Each valid triangle yields six defined products

$$ab = c \quad a^{-1}c = b \quad bc^{-1} = a^{-1} \quad \cdots$$

The geometric meaning of the associative law: if three valid triangles fit around a common vertex, then their perimeter is valid too.
Embeddability

The universal group, $U(P)$, of a pree, $P$, has generators $P$ and relations $ab = c$ for all products defined by the partial multiplication.

The question of whether a finite pree embeds in its universal group is undecidable. [T. Evans 1951]

Baer introduced several axioms sufficient for embeddability, Stallings[1971] used Axiom P5 to define pregroups.

Several other axioms have been considered. See [Gaglione, Lipschutz, Spellman, 2012].

![Axiom P5](image1.png)

![Axiom S](image2.png)
Theorem

Universal groups of finite prees satisfying Axiom P5 or Axiom S are virtually free.

Apply the characterization of virtually free groups as groups with context free word problem [Muller, Schupp 1983]

We consider prees satisfying two axioms introduced by Baer and illustrated below.

Axioms A4 and A5
**Theorem (RG 2014)**

*If* $P$ *is a pree satisfying Axioms A(4) and A(5), then* $P$ *embeds in* $U(P)$, *and the multiplication in* $P$ *is induced by the multiplication in* $U(P)$.

The Cayley diagram of a pree has the elements of $P - \{1\}$ as its vertices and edges corresponding to valid products $xy = z$ with $x, y, z \in P - \{1\}$. Each such product determines an edge from $x$ to $z$ with label $y$.

The Cayley diagram is essentially the link of the vertex of the presentation complex corresponding to the pree.

Axioms A4 and A5 are a weaker form to the small cancellation condition C3-T6.

There are many such variations by e.g., A. Juhasz, J. McCammond and D. Wise, Yu. Ol’shanskii, U. Weiss.
Theorem (RG, 2014)

If $P$ is a finite pree satisfying Axioms A(4) and A(5), then $U(P)$ is biautomatic.

The theorem generalizes [Gersten and Short 1990] for $C(3)$-$T(6)$ small cancellation presentations with all pieces of length 1.

Finite $C(3)$-$T(6)$ groups are cyclic, but all finite groups are universal groups of $A4$-$A5$ prees (namely themselves).
**Theorem (RG)**

Let $P$ satisfy A4 and A5. The universal group of $P$ has a regular geodesically perfect rewriting system.

[Diekert, Duncan, Miasnikov, 2010] extend the Knuth-Bendix procedure to a procedure for geodesically perfect rewriting system as follows

A geodesically perfect rewriting system contains

1. Length-reducing reductions which rewrite any word to an equivalent geodesic word;
2. Length-preserving reductions which rewrite any two equivalent geodesic words to each other.
Example 1.

\[ G = \langle a, b \mid ab = ba \rangle \]

Define a pree \( P \) by triangulating the relations.

\[ ab = c = ba \]

\( G \) is the universal group of the pree \( P = \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\} \) and all products corresponding to the relations above.

Check associativity and the two additional axioms by looking for cycles of length at most 6 in the Cayley diagram of \( P \).

The geodesically perfect rewriting system is regular and infinite.
Example 2. $G = \langle a, b, t \mid [a, t] = [b, t] = 1 \rangle$
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The triangles determine a pree.
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\[
\begin{array}{cccccc}
\text{a} & \text{a}^{-1} & \text{b} & \text{b}^{-1} & \text{t} & \text{t}^{-1} \\
\text{a} & 1 & \text{b} & \text{b}^{-1} & \text{t} & \text{t}^{-1} \\
\text{a}^{-1} & 1 & \text{b} & \text{b}^{-1} & \text{t} & \text{t}^{-1} \\
\text{b} & 1 & \text{x} & \text{x}^{-1} & \text{t} & \text{t}^{-1} \\
\text{b}^{-1} & 1 & \text{x} & \text{x}^{-1} & \text{t} & \text{t}^{-1} \\
\text{t} & \text{x} & \text{x}^{-1} & \text{y} & \text{y}^{-1} & 1 \\
\text{t}^{-1} & \text{x}^{-1} & \text{y}^{-1} & 1 & \text{a} & \text{a}^{-1} \\
\text{x} & \text{t} & \text{a}^{-1} & \text{a} & 1 \\
\text{x}^{-1} & \text{t}^{-1} & \text{a}^{-1} & \text{a} & 1 \\
\text{y} & \text{t}^{-1} & \text{b} & \text{b}^{-1} & 1 \\
\text{y}^{-1} & \text{t}^{-1} & \text{b} & \text{b}^{-1} & 1 \\
\end{array}
\]

The triangles determine a pree.
Example 2. $G = \langle a, b, t \mid [a, t] = [b, t] = 1 \rangle$

$G$ is biautomatic if there are no cycles of length $< 6$ in the Cayley diagram of the partial multiplication table.
One-relator groups

Conjecture. Every one-relator group whose relator is a commutator is automatic.

Example 3. \( G = \langle a, b, c, d, e \mid [ae, bcabd] = 1 \rangle \)

It suffices that \( G * F_3 = \langle a, b, c, d, e, p, q, r \mid [ae, bcabd] = 1 \rangle \) is automatic.
Triangles of groups: Gersten and Stallings, Corson, Bridson and Haefliger.

Non-positively curved triangles with finite vertex groups have biautomatic universal groups [Floyd and Parry 1995].

Tits Alternative: [Howie, Kopteva 2006], [Cuno, Lehnert 2014]

Positively curved triangles: [Chermak 1995], [Allcock 2012]

A triangle of groups is a pree.
Triangles of prees

**Theorem (RG)**

Let $T$ be a triangle of prees satisfying A4 - A5 and such that edge prees have the induced multiplication from each of their vertex prees and satisfy the following conditions.

1. The distance between any two nontrivial elements of an edge pree is $\geq 3$ in each of its vertex prees.

2. The distance between any two nontrivial elements in distinct edge prees of a vertex pree is not 2.

Then $T$ also satisfies A4-A5.
Example 4

$F_{a,b}$

$\langle a \rangle$  \hspace{1cm} $\langle b \rangle$

$\langle c \rangle$

$F_{a,c}$  \hspace{1cm} $F_{a,c}$

$F_{x,y}$ is the universal group of the pre-order $P_{x,y} = \{1, x, x^{-1}, y, y^{-1}\}$ with no products defined.

$P_{a,b}$

$\{1, a, a^{-1}\}$  \hspace{1cm} $\{1, b, b^{-1}\}$

$\{1, c, c^{-1}\}$

$P_{a,c}$  \hspace{1cm} $P_{a,c}$
Example 5

Vertex groups $\langle a, b, c \mid a^b = [a, c] \rangle$

Edge prees: $\{1, a, a^{-1}\}$ and $\{1, b, b^{-1}\}$.

The triangle of universal groups of edge and vertex prees has all angles 0. The universal group of the triangle is biautomatic but not hyperbolic.