CHAPTER 3. INTRODUCTION TO MATRIX METHODS FOR STRUCTURAL ANALYSIS

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Modern methods of structural analysis overcome some of the drawbacks of classical techniques. Statically determinate and indeterminate structures are solved the same way. Trusses, beams or frames are solved with the same method. The method can be automated so we can use computers to solve.

Structural analysis using the matrix method does not involve any new concepts of structural engineering or mechanics.

All linear elastic structures (statically determinate or indeterminate) are governed by systems of linear equations. Below are some important definitions needed for creating the system of equations.

DEFINITIONS

Elements: The individual members that comprise a structure.

- **Truss Element:** A member subjected to axial forces only. Shown here is the type of force and the associated axial deformation assigned on a truss element.

- **Beam Element:** A member that resists transverse forces and moments. Shown here are the shear and moment forces and associated transverse and rotational displacements assigned on a beam element.
- **Frame Element**: A member that resists axial forces in addition to moments and transverse shear. Shown here are the forces and associated axial, transverse and rotational displacements assigned on a frame element.

- **Degrees Of Freedom (DOF)**: A possible displacement (translational or rotational) associated with the forces on the different types of elements.

- **Nodes**: Nodes are locations in the structure where elements are connected.

- **Stiffness**: Force required to induce a unit deformation in an elastic material. You can also think of it as the material’s resistance to deformation. Take for example a spring. If it takes a force of 10lbs to deform the spring by one inch, then the spring stiffness, $k$, is 10 lb/in.
**Coordinates**

- **Global Coordinates:** Identify (with a unique number assigned) each DOF of the structure in the global coordinates.

  ➢ Coordinates are assigned to each DOF for each node
  ➢ Coordinates govern entire structure, not individual members
  ➢ Global coordinates have a unique number for each DOF of the entire structure

- **Local Coordinates:** Identify DOF at the element level

For example, local coordinates 4, 5, 6 of the frame element correspond to global coordinates 11, 10, 12 of the frame.
CREATING THE ELEMENT STIFFNESS MATRIX FOR A SPRING

An element with \( n \) degrees of freedom is associated with an \( n \times n \) stiffness matrix. For example, a spring has two degrees of freedom (truss element). As such, the element stiffness matrix is a \( 2 \times 2 \) matrix. In what follows, we will use the definition for a stiffness coefficient, \( K_{ij} \), to produce the element stiffness matrix for a spring.

\[
K_e = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\]

**Definition 1:** The stiffness coefficient \( K_{ij} \) is the force at coordinate \( i \) when a unit displacement is imposed at coordinate \( j \), and the displacements at all other coordinates are constrained to zero (fixed).

Creating the stiffness matrix of a spring

Consider the spring in the figure. The spring constant, \( k = 10 \text{ lb/in} \) is the force required to cause a unit displacement: \( f = kx \). Here, a force of 10 lb placed on node 2, causes a unit displacement at node 2. At the same time, a reaction of 10 lbs develops at node 1 to keep the system in static equilibrium and to ensure that the displacement of node 1 is zero.

If we consider *Definition 1*, we see that we defined \( K_{22} \) as a force at coordinate 2 when a unit displacement is imposed at coordinate 2. Therefore, \( K_{22} \) has a value of 10. We also defined \( K_{12} \) as the force that develops at coordinate 1 when a unit displacement is imposed at coordinate 2. Therefore, \( K_{12} \) has the value of \(-10\). These values are inserted into the Stiffness Matrix for the element as follows:

\[
K_e = \begin{bmatrix}
K_{11} & -10 \\
K_{21} & 10
\end{bmatrix}
\]
As seen, node 2 was used to fill-in column two of the stiffness matrix. Now, to fill-in column one, we will work with the release of node 1 while fixing node 2.

A 10-lb force is applied at node 1, imposing a unit displacement on node 1. This force is equivalent to \( K_{11} \). The reaction force of -10 lbs that develops at Node 1 is equivalent to \( K_{21} \). Completing the remainder of the stiffness matrix, we obtain:

\[
K_e = \begin{bmatrix}
10 & -10 \\
-10 & 10
\end{bmatrix}
\]

Note that the stiffness of the bar is 10 k/in. The stiffness matrix, \( K \), can be rewritten as:

\[
K_e = 10 \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

In general, the stiffness matrix for a single spring element of stiffness \( k \) can be written as:

\[
K_e = k \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

Now assume that the spring is loaded as shown below. We can solve for the unknown displacement and force using the relationship \( \{f\} = [K]\{x\} \), where \([K]\) is given above.

\[
\begin{bmatrix}
200 \\
F_2
\end{bmatrix} = \begin{bmatrix}
10 & -10 \\
-10 & 10
\end{bmatrix} \begin{bmatrix}
x_1 \\
0
\end{bmatrix}
\]

Solving the system of equations, we get: \( x_1 = 20 \) in, \( F_2 = -200 \) lb.
CREATING THE STIFFNESS MATRIX OF A STRUCTURE OF INTER-CONNECTED SPRINGS

Let's investigate a system made up of three springs.

![Diagram of three interconnected springs](image)

Let's investigate a system made up of three springs.

○ = Node
□ = Element
→ = Degree of Freedom

The structural stiffness matrix is of size $(4 \times 4)$ since we are dealing with four degrees of freedom. It is developed one column at a time. To find the first column:

- Force node 1 one unit to the right
- Fix all other nodes
- Calculate the stiffness coefficients $K_{ij}$ by finding the reactions forces that satisfy equilibrium.

Namely:

$\sum F@node1: \ K_{11} - 10 - 20 = 0; \quad K_{11} = 30$

$\sum F@node2: \ K_{21} + 20 = 0; \quad K_{21} = -20$

$\sum F@node3: \ K_{31} + 10 = 0; \quad K_{31} = -10$
The first column is written as:

\[
\begin{bmatrix}
30 \\
-20 \\
-10 \\
0
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 2 to find the second column. Namely:

\[\sum F@1: \ K_{12} + 20 = 0; \quad K_{12} = -20\]
\[\sum F@2: \ K_{22} - 30 - 40 = 0; \quad K_{22} = 50\]

\[\sum F@4: \ K_{42} + 30 = 0; \quad K_{32} = -30\]

Substituting into the stiffness matrix, we obtain:

\[
\begin{bmatrix}
30 & -20 \\
-20 & 50 \\
-10 & 0 \\
0 & -30
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 3 to find the third column. Namely:

\[K_{13} = -10\]
\[K_{23} = 0\]
\[K_{33} = 10\]
\[K_{43} = 0\]

Repeat the calculations after fixing all nodes except node 4 to find the fourth column.

\[
\begin{bmatrix}
30 & -20 & -10 \\
-20 & 50 & 0 \\
-10 & 0 & 10 \\
0 & -30 & 0
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 4 to find the fourth column. Namely:

\[K_{14} = 0\]
\[K_{24} = -30\]
\[K_{34} = 0\]
\[K_{44} = 30\]

Substituting into the Stiffness Matrix:
Now assume that the spring is loaded as shown below. We can solve for the unknown displacement and force using the relationship \( \{f\} = [K] \{x\} \), where \([K]\) is given above.

Solution

Known: \( x_3 = 0 \)  
\( x_4 = 0 \)  
\( F_1 = 40 \) lb  
\( F_2 = 100 \) lb

Unknown: \( x_1 = ? \)  
\( x_2 = ? \)  
\( F_3 = ? \)  
\( F_4 = ? \)

Using the basic equation \( \{F\} = [K] \{x\} \):

\[
\begin{bmatrix}
40 \\
100 \\
F_3 \\
F_4
\end{bmatrix}
= 
\begin{bmatrix}
30 & -20 & -10 & 0 \\
-20 & 50 & 0 & -30 \\
-10 & 0 & 10 & 0 \\
0 & -30 & 0 & 30
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
0 \\
0
\end{bmatrix}
\]

This matrix equation cannot be solved as is stands. It must divided into parts where only forces or displacements are the only unknowns. Also, since \( x_3 \) and \( x_4 \) are equal to zero, columns 3 and 4 can be eliminated from the equations.

This leaves us with the reduced matrix equation:
And we can solve for the unknown displacements $x$.

% Solving a set of linear equations
k=[30 -20; -20 50];
d=[40 100];
x=k\d'
x =
  3.6364
  3.4545

Once all the displacements are known, they can be used to solve for the remaining forces.

% Solving for the reactions
k=[-10 0; 0 -30];
d=[3.636 3.454];
F=k * d'
F =
  -36.3600
  -103.6200
Here we will solve the same three-spring system by using the energy expressions.

The strain energy $U$ for the structure can be expressed as:

$$U = \frac{1}{2} k_1 (x_1-x_3)^2 + \frac{1}{2} k_2 (x_2-x_1)^2 + \frac{1}{2} k_3 (x_4-x_2)^2$$

where $x_i$ is the displacement of every node $i$. The potential energy $P$ can be expressed as:

$$P = F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4$$

The total energy $E$ in the system at any time is $E(x) = U - P$. In view of the equations above, this relationship can be written as:

$$E(x) = \frac{1}{2} x^T K x - x^T F$$  \hspace{1cm} (equation 1)

where

$$K = \begin{bmatrix}
  k_1 + k_2 & -k_2 & -k_1 & 0 \\
  -k_2 & k_2 + k_3 & 0 & -k_3 \\
  -k_1 & 0 & k_1 & 0 \\
  0 & -k_3 & 0 & k_3
\end{bmatrix}$$

**Note:** In order to understand how to work with matrices Please expand Equation 1 using the $K$ matrix above and show that it produces the energy of the structure $U-P$.

After the loading $F$ is applied to the system, the system deforms. In this case, each DOF $i$ displaces an amount $x_i$. We can solve for these displacements by solving for the vector $x$ that minimizes the energy of the system. Taking the derivative of the energy with the respect to $x$ and setting this derivative to zero:

$$\frac{\partial}{\partial x} \left[ \frac{1}{2} x^T K x - x^T F \right] = 0$$

we arrive at:

$$K x = F$$
The solution of this equation for $x$ gives the displacement that each DOF has undergone to help the system achieve equilibrium. In other words, under loading, the system will deform to a configuration that minimizes the total energy of the system. This is its equilibrium position.

**Note:** Another way to arrive at the $Kx=F$ is to minimize the energy wrt $x_1$, $x_2$, $x_3$, and $x_4$. Then write the four resulting equations in matrix form.
HOMEWORK 6—STIFFNESS MATRIX OF CONNECTED SPRINGS USING NEWTON AND ENERGY METHODS.

1) Repeat the lecture on energy minimization. Solve for all with your MATLAB (do not turn this in)

2) Given that $k_1=10$ and $k_2=30\text{lb/in}$ for the spring below.
   - Create the stiffness matrix of the structure by freeing one degree-of-freedom at a time while constraining all others to zero.
   - Create the stiffness matrix for the structure by writing the kinetic and potential energy expressions for the springs and minimizing the energy.
   - Apply boundary conditions and loading (assume a force $F_1=-30$ is applied on node one.) Solve for the equilibrium position.
   - Plot the energy of the system in MATLAB (note that you must apply boundary conditions and loading before you solve for the energy.)
   - Show graphically that the vector $\mathbf{x}$ that minimizes the energy of the system is the same as the one that solves the equations $K\mathbf{x}=\mathbf{F}$
   - Use a MATLAB optimization function to find the minimum of the energy. Show that minimization of energy gives you the same displacements as the solution of Newton's Equations $K\mathbf{x}=\mathbf{f}$. 

![Diagram of connected springs](image-url)