FE610 Stochastic Calculus for Financial Engineers
Lecture 1. Introduction

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Outline

Instructor: Dr. Steve Yang, Babbio 536,
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- **Class Time:** Lectures on Thursday 03:15PM-05:30PM
  $01 - 14 - 201305 - 15 - 2013$
- **Office Hours:** Wednesday 10:00AM-11:00AM at Babbio
  536
- **Prerequisites:** N/A

- **Topics:**
  This course provides the mathematical foundation for
  understanding modern financial theory. It includes topics
  such as basic probability theory, random variables, discrete
  and continuous distributions, Martingale processes,
  Brownian motion, stochastic integration and Ito process
  and calculus. Applications to financial concepts and
  instruments are discussed throughout the course.

- **Textbooks:**
    - **"Introduction to the Mathemtics of Financial Derivatives" by Salih N Neftci, 2nd ed, AP ISBN 0125153929 [REQUIRED]**
      Salih N. Nefti (14 July 1947 − 15 April 2009) was a leading expert in the fields of financial markets and financial engineering. He served many advisory roles in national and international financial institutions, and was an active researcher in the fields of finance and financial engineering. Professor Nefti was an avid and highly regarded educator in mathematical finance who was well known for a lucid and accessible approach towards the field.

    - "Stochastic Calculus and Financial Applications", by J. Michael Steele, Springer 2000, ISBN-10: 0387950168, ISBN-13: 978-0387950167 [OPTIONAL]
    - "Financial Calculus" by Martin Baxter and Andrew Rennie, Cambridge University Press, 1999. ISBN 0 521 55289 3. [OPTIONAL]

Policies

Homework Honor Policy:

- You are allowed to discuss the problems between yourselves, but once you begin writing up your solution, you must do so independently, and cannot show one another any parts of your written solutions. The homework is to be pledged (for undergraduate students).

- Your solutions to the homework and exam problems have to be typed (written legibly) and uploaded to the Moodle course website in one single PDF file (no other file format will be accepted). Any changes to the course schedule or due date of assignments will be announced through the course website.

- Each homework assignment will contain 3-5 problems, and will be posted on the class website. No late homework will be accepted under any circumstances.

Exams & Grades

Grades: Homework Assignments - 40%; Mid-term - 30%; Final - 30%.

Exams: Two Exams. (Mid-term) EXAM I: March 7 - (Thursday). (Final) EXAM II: May 9 - (Thursday). These exams will consist of short questions, and mathematical problems.

Exam must be taken at these times No Exceptions!!!!!!!

Financial Derivatives - A Derivative Instrument

DEFINITIONS: A financial contract is a *derivative security*, or a *contingent claim*, if its value at expiration date T is determined *exactly* by the market price of the underlying cash instrument at time T (Ingersoll, 1987).

- At the time of the expiration of the derivative contract, denoted by T, the price $F(T)$ of a derivative asset is completely determined by S_T, the value of the "underlying asset." After that date, the security ceases to exist.

- In the rest of the course, we will use symboles $F(t)$ and $F(S_t, t)$ alternately to denote the price of a derivative product written on the underlying asset S_t at time t. The financial derivative is sometimes assumed to yield a payout d_t. At other times, the payout is zero. T will always denote the expiration date.

Financial Derivatives - Types of Derivatives

Three Types: Futures and forwards, Options and Swaps.

Forwards and options are considered *basic building blocks*. Swaps and some
other complicated structures can eventually be decomposed into sets of basic
forwards and options.

- We have five main groups of the *underlying securities*:
    - **Stocks**: These are claims to "real" returns generated in the
      production sector for goods and services.
    - **Currencies**: These are liabilities of governments or banks.
    - **Interest rates**: Not assets, but a notional asset that one can
      take a position on the direction of future interest rates.
    - **Indexes**: Not assets, but derivative contracts can be written
      on notional amounts and a position can be taken with respect
      to the direction of the underlying index.
    - **Commodities**: Soft commodities (cocoa, coffee, and sugar),
      Grans (barley, corn, cotton, soybean, etc.), Metals (copper,
      nickel, tin, and others), Energy (crude oil, fuel oil, etc), ...

Another Classification of Derivatives

- **Cash-and-Carry Markets**: Some derivative instruments are written on products of *cash-and-carry* markets. Gold, silver, currencies, and T-bonds are some examples of cash-and-carry products.

  In these markets, one can borrow at risk-free rates (by collateralizing the underlying physical asset), buy and store the product, and insure it until the expiration date of any derivative contract. One can therefore easily build an alternative to holding a forward or futures contract on these commodities.

  Information about future demand and supplies of the underlying instrument should not influence the "spread" between cash and futures (forward) prices. After all, this spread will depend mostly on the level of risk-free interest rates, storage, and demands of the underlying instrument is expected to make the cash price and the future price change by the same amount.

Another Classification of Derivatives

- **Price-Discovery Markets**: Here, it is physically impossible to buy the underlying instrument for cash and store it until some future expiration date. Such goods either are too *perishable* to be stored or may not have a cash market at the time the derivative is trading.

    One example is a contract on spring wheat. When the future contract for this commodity is traded in the exchange, the corresponding cash market may not yet exits. future interest rates.

    The strategy of borrowing, buying, and storing the asset until some later expiration date is not applicable to price-discovery markets. Under these conditions, any information about the future supply and demand of the underlying commodity cannot influence the corresponding cash price. Such information can be discovered in the futures market, hence the terminology.

Another Classification of Derivatives

- **Expiration Date**: The relationship between $F(t)$, the price of
  the derivative, and S_t, the value of the underlying asset, is
  known exactly (or deterministically), only at the expiration
  date T. In the case of forwards or futures, we expect:

$$F(T) = S_T; \tag{1}$$

  For example, the (exchange-traded) futures contract promising
  the delivery of 100 troy ounces of gold cannot have a value
  different from the actual market value of 100 troy ounces of
  gold on the expiration date of the contract. They both
  represent the same thing at time T. So, in the case of gold
  futures, we can indeed say that the equality in the last
  equation holds at expiration.
  At $t < T$, $F(T)$ may not equal S_t. Yet we can determine a
  function that ties S_t to $F(T)$.

Forwards

- **DEFINITION**: A forward contract is an obligation to buy (sell) an underlying asset at a specified forward price on a known date.
    - The expiration date of the contract and the forward price are written when the contract is entered into. If a forward purchase is made, the holder of such a contract is said to be *long* in the underlying asset. If at expiration the cash price is higher than the forward price, the long position makes a profit; otherwise there is a loss.
    - Figure 1: The contract is purchased for $F(t)$ at time t. It is assumed that the contract expires at time $t + 1$. The upward-sloping line indicates the profit or loss of the purchaser at expiration. The slope of the line is one.
    - If S_{t+1} exceeds $F(t)$, then the long position ends up with a profit. Given that the line has unitary slop, the segment AB equals the vertical line BC.
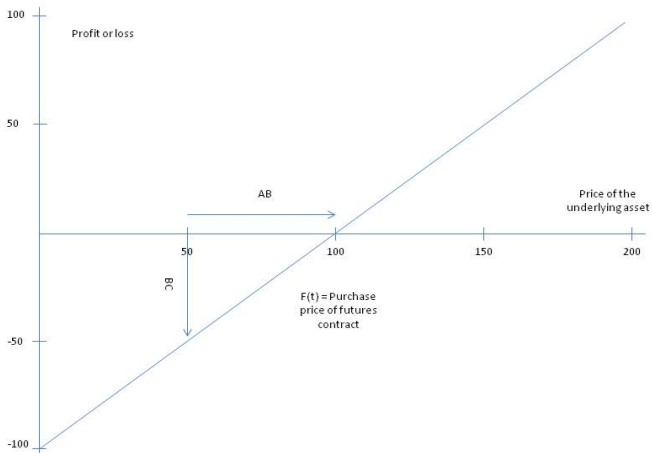   - *Futures and forwards are linear instruments.*
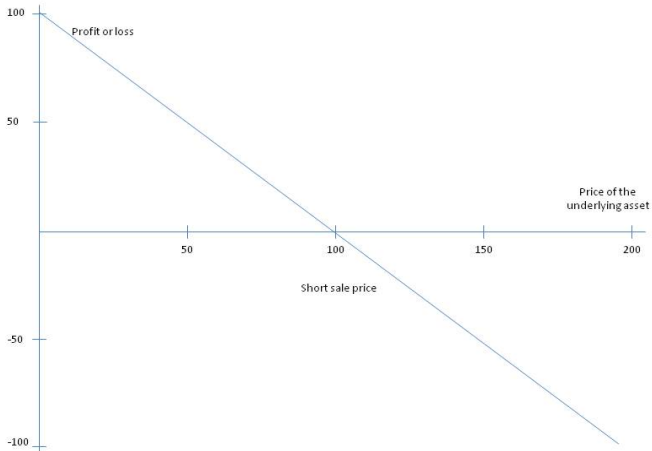
Figure : Payoff diagram of a simplified long position.

Figure : Payoff diagram of a simplified short position.

Futures

> **Futures** and **forwards** are similar instruments. The major differences can be stated briefly as follows:
>
> - Futures are traded in formalized exchanges. The exchange designs a standard contract and sets some specific expiration dates. Forwards are custom-made and are traded over-the-counter.
>
> - Futures exchanges are cleared through exchange clearing houses, and there is an intricate mechanism designed to reduce the default risk.
>
> - Futures contracts are marked to market. That is, every day the contract is settled and simultaneously a new contract is written. Any profit or loss during the day is recorded accordingly in the account of the contract holder.

Options (European vs. American Options)

> **DEFINITION:** A European-type call option on a security S_t
> is the right to buy the security at a preset *strike price* K. This
> right may be exercised *at the expiration date* T of the option.
> The call option can be purchased for a price of C_t dollars,
> called the premium, at time $t < T$.

- American options can be exercised any time between the
  writing and the expiration of the contract.

- There are several reasons that traders and investors may want
  to calculate the arbitrage-free price, C_t of a call option.
  Before the option is first written at time t, C_t is not known.
  A trader may want to obtain some estimate of what this price
  will be if the option is written. If the option is an
  exchange-traded security, it will start trading and a market
  price will emerge. If the option trades over-the-counter, it
  may also trade heavily and a price can be observed.

Option Pricing

- At time t, the only know formula concerning C_t is the one that determines its value at the time of expiration T. Assuming:

  if there is no commissions and/or fees

  if the bid-ask spread on S_t and C_t are zero,

  then at expiration, C_T can assume only two possible values.

  1. The option is expiring *out-of-money*:

  $$S_T < K \qquad (2)$$

  2. The option is expiring *in-the-money*:

  $$S_T > K \qquad (3)$$

  We can use a shorthand notation to express both of these possibilities by writing:
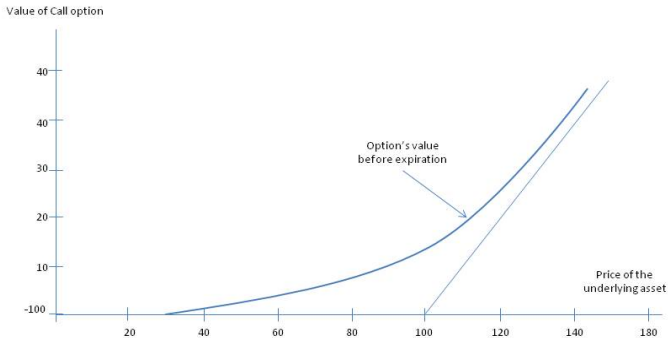
  $$C_T = max[S_T - K, 0] \qquad (4)$$

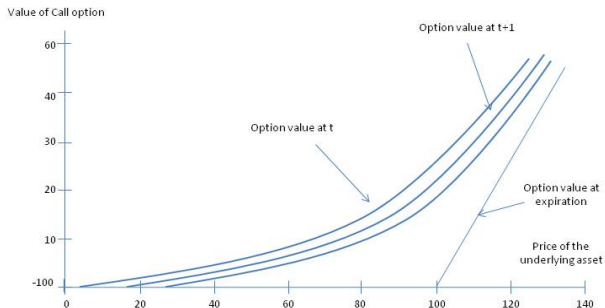Figure : Call Option Relationship between S_T and C_T

Figure : Call Option Value before Expiration

Swaps

> **DEFINITION**: A swap is the simultaneous selling and purchasing of cash flows involving various currencies, interest rates, and a number of other financial assets.
>
> - Swaps and swoptions are among some of the most common types of derivatives. One method for pricing swaps and swoptions is to *decompose* them into forwards and options.
>
> - Decomposing a swap into its constituent components is a potent example of financial engineering and derivative asset pricing. It also illustrates the special role played by simple forwards and options.
>
> - It is always possible to decompose simple swap deals into a basket of simpler forward contracts. The basket will replicate the swap. The forward can then be priced separately, and the corresponding value of the swap can be determined from these numbers.

Example

An interest rate swap between two counterparties A and B is created as a result of the following steps:

1. Counterparty A needs a $1 million floating-rate loan. B needs a $1 million fixed-rate loan. But because of market conditions and their relationships with various banks, B has a comparative advantage in borrowing at a floating rate.

2. A and B decided to exploit this comparative advantage.

3. Counterparty A borrows $1 million at a fixed rate. The interest payments will be received from counterparty B and paid back to the lending bank.

4. Counterparty B borrows $1 million at the floating rate. Interest payments will be received from counterparty A and will be repaid to the lending bank.

5. Note that the initial sums, each being $1 million, are identical. Hence, they do not have to be exchanged. they are called *notional principals*. The interest payments are also in the same currency. Hence, the counterparties exchange only the interest differentials.