FE670 Algorithmic Trading Strategies
Lecture 6. Portfolio Optimization: Basic Theory and Practice

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Outline

Mean-Variance Analysis: Overview

- Markowitz's starting point is that of a rational investor who, at time t, decides what portfolio of investments to hold for a time horizon of δt.

- The investor makes decisions on the gains and losses he will make at time $t + \delta t$.

- At time $t + \delta t$, the investor will reconsider the situation and decide anew.

* This one-period framework is often referred to as *myopic or short-sighted* behavior. In general, a myopic investor's behavior is suboptimal in comparison to an investor who makes investment decisions based upon multiple periods ahead.
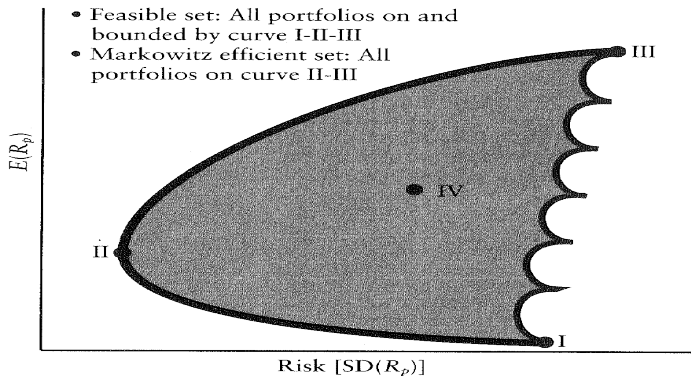
- Markowitz reasoned that investors should decide on the basis
  of a trade off between risk and expected return. Expected
  return of a security is defined as the expected price change
  plus any additional income over the time horizon considered,
  such as dividend payments, divided by the beginning price of
  the security.

- He suggested that risk should be measured by the variance of
  returns - the average squared deviation around the expected
  return.

* Markowitz's mean-variance framework does not assume joint
  normality of security returns. However, the mean-variance
  approach is consistent with two different starting points:

  (1) expected utility maximization under certain assumptions.
  (2) the assumption that security returns are jointly normally
  distributed.

- Exhibit 8.1 provides a graphical illustration of the efficient frontier.

EXHIBIT 8.1   Feasible and Markowitz Efficient Portfolios

- Feasible set: All portfolios on and bounded by curve I-II-III
- Markowitz efficient set: All portfolios on curve II-III

$E(R_p)$

Risk [SD(R_p)]

- The set of all possible portfolios that can be constructed are called *feasible set*. The set of all mean-variance efficient portfolios, for different desired returns, is called the efficient frontier.

- Moreover, Markowitz argued that for any given level of expected return, a rational investor would choose the portfolio with minimum variance from amongst the set of all possible portfolios.

- Therefore, the efficient frontier provides the best possible trade-off between expected return and risk. The portfolio at point II is often referred to as the *global minimum variance portfolio (GMV)*, as it is the portfolio on the efficient frontier with the smallest variance.

- Exhibit 8.2 shows the investment process often referred to as *mean-variance optimization or theory of portfolio selection*.

EXHIBIT 8.2 The MPT Investment Process
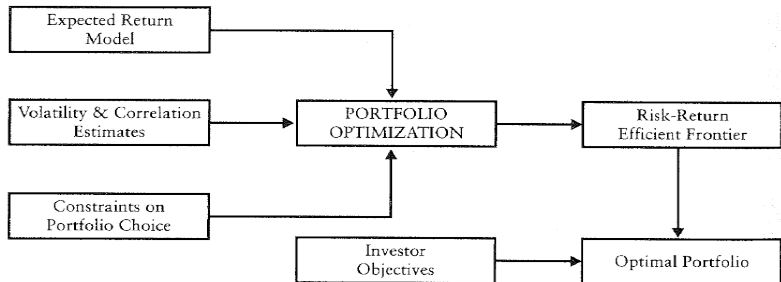
Source: Exhibit 2 in Frank J. Fabozzi, Francis Gupta, and Harry M. Markowitz, "The Legacy of Modern Portfolio Theory," *Journal of Investing*, 11 (Fall 2002),

Classical Framework for Mean-Variance Optimization

- Suppose an investor has to choose a portfolio comprised of N risky assets. The investor's choice is embodied in an N-vector $\mathbf{w} = (w_1, w_2, ..., w_N)'$ of weights, where each weight i represents the fraction of the i-th asset held in the portfolio

$$\sum_{i=1}^{N} w_i = 1$$

  For now, we permit short selling, which means that weights can be negative.

- Suppose the assets' returns $\mathbf{R} = (R_1, R_2, ..., R_N)'$ have expected returns $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_N)'$ and covariance matrix given by

$$\Sigma = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \dots & \sigma_{NN} \end{bmatrix}$$

  where σ_{ij} denotes the covariance between asset i and j.

- Under these assumptions, the return of a portfolio with
  weights **w** is a random variable $R_p = \mathbf{w'R}$ with expected
  return and variance given by

$$\mu_p = \mathbf{w'}\boldsymbol{\mu}$$
$$\sigma_p^2 = \mathbf{w'}\boldsymbol{\Sigma}\mathbf{w}$$

  For now, we simply assume that expected returns, $\boldsymbol{\mu}$, and
  their covariance matrix, $\boldsymbol{\Sigma}$, are given.
- To calculate the weights fro one possible pair, we choose a
  targeted mean return, μ_0, Following Markowitz, the investor's
  problem is constrained minimization problem:

$$\min_w \frac{1}{2}\mathbf{w'}\boldsymbol{\Sigma}\mathbf{w}$$
$$\text{s. t.}$$
$$\mu_0 = \mathbf{w'}\boldsymbol{\mu}, \mathbf{w'l'} = 1, \mathbf{l'} = [1, 1, ..., 1]$$

- We refer to this version of the classical mean-variance
  optimization problem as the *risk minimization formulation*.

Mean-Variance Analysis: Overview **Classical Framework for Mean-Variance Optimization** Mean-Variance Optimization with a Risk-F

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- This problem is a quadratic optimization problem with equality constraints with the solution given by

$$w = \lambda \Sigma^{-1} I + \gamma \Sigma^{-1} \mu$$

where

$$\lambda = \frac{C - \mu_0 B}{\Delta}, \gamma = \frac{\mu_0 A - B}{\Delta}$$

$$A = I' \Sigma^{-1} I, B = I' \Sigma^{-1} \mu, C = \mu' \Sigma^{-1} \mu$$

- It is easy to see that

$$\sigma_0^2 = \mathbf{w}' \Sigma \mathbf{w}$$

$$= \frac{A\mu_0^2 - 2B\mu_0 + C}{\Delta}$$

* In extensions involving only so-called equality constraints, finding the optimum portfolio reduces to solving a set of linear equations. For formulations involving inequality constraints, analytical solutions are not available, and numerical optimization needs to be applied.

- The mean-variance optimization problem has several alternative but equivalent formulations that are very useful in practical applications.
- First, we can choose a certain level of targeted portfolio risk, say σ_0, and then maximize the expected return of the portfolio (expected return maximization formulation):

$$\max_w \left[w' \boldsymbol{\mu} \right]$$
$$\text{s. t.}$$
$$\sigma_0^2 = \mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}, \mathbf{w}' \mathbf{l}' = 1, \mathbf{l}' = [1, 1, ..., 1]$$

- Alternatively, we can explicitly model the trade-off between risk and return in the objective function using a risk-aversion coefficient λ (risk-aversion formulation):

$$\max_w \left[w' \boldsymbol{\mu} - \lambda \frac{1}{2} \mathbf{w}' \boldsymbol{\Sigma} \mathbf{w} \right]$$
$$\text{s. t.}$$
$$\mathbf{w}' \mathbf{l}' = 1, \mathbf{l}' = [1, 1, ..., 1]$$

Mean-Variance Optimization with a Risk-Free Asset

- Assume that there is risk-free asset, with risk-free return denoted by R_f and that the investor is able to borrow and lend at this rate. The investor has to choose a combination of the N risky assets plus the risk-free asset.

- The weights $\mathbf{w}'_R = (w_{R1}, w_{R2}, ..., w_{RN})$ do not have to sum to 1 as the remaining part $(1 - \mathbf{w}'\mathbf{l})$ is the investment in the risk-free asset.

- The portfolio's expected return and variance are

$$\mu_p = \mathbf{w}'_R \boldsymbol{\mu} + (1 - \mathbf{w}'_R l) R_f$$
$$\sigma_p^2 = \mathbf{w}'_R \Sigma w_R$$

because the risk-free asset has zero variance and is uncorrelated with the risky assets.

- The investor's objective is again for a targeted level of
  expected portfolio return, μ_0, to choose allocations by solving
  a quadratic optimization problem

$$\min_{w_R} \mathbf{w}'_R \Sigma w_R$$
subject to
$$\mu_0 = \mathbf{w}'_R \boldsymbol{\mu} + (1 - \mathbf{w}'_R l) R_f$$

- The optimal portfolio weights are given by

$$\mathbf{w}_R = C \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - R_f \mathbf{l})$$
where
$$C = \frac{\mu_0 - R_f}{(\boldsymbol{\mu} - R_f \mathbf{l})' \Sigma^{-1}(\boldsymbol{\mu} - R_f \mathbf{l})}$$

Therefore, with a risk-free asset, all minimum variance
portfolios are a combination of the risk-free asset and a given
risky portfolio. The risky portfolio is often called *tangency
portfolio* or *market portfolio*.

- Exhibit 8.3 Capital Market Line and the Markowitz
  Efficient Frontier.

EXHIBIT 8.3   Capital Market Line and the Markowitz Efficient Frontier
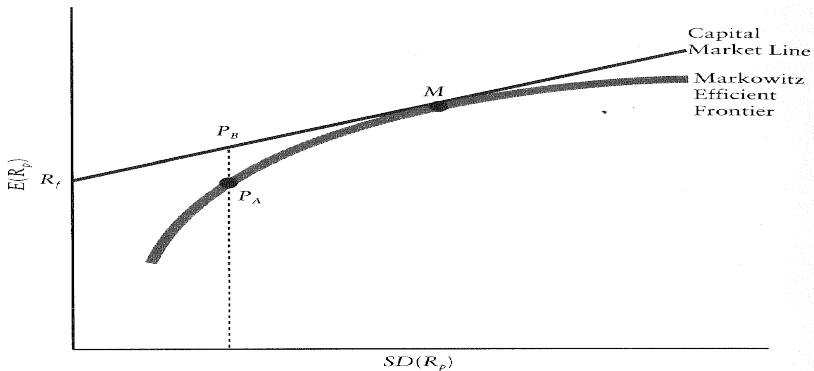
- In Exhibit 8.3 every combination of the risk-free asset and the market portfolio *M* is shown on the line drawn from the vertical axis at the risk-free rate tangent to the Markowitz efficient frontier. All the portfolios on the line are feasible for the investor to construct. The line from the risk-free rate that is tangent to the efficient frontier of risky assets is called the *Capital Market Line* (CML).

  We observe that with the exception of the market portfolio, the minimum variance portfolios that are a combination of the market portfolio and the risk-free asset are superior to the portfolio on the Markowitz efficient frontier for the same level of risk.

- With the introduction of the risk-free asset, we can now say that an investor will select a portfolio on the CML that represents a combination of borrowing or lending at the risk-free rate and the market portfolio. This important property is called *separation*.

- Portfolio to the left of the market portfolio represent combinations of risky assets and the risk-free asset.

- Portfolio to the right of the market portfolio include purchases of risky assets made with funds borrowed at the risk-free rate. Such a portfolio is called a *leveraged portfolio* because it involves the used of borrowed funds.

- Practical portfolio construction is normally broken down into at least two steps:

  1. Asset allocation: Decide how to allocate the investor's wealth between the risk-free security and the set of risky securities

  2. Risky portfolio construction: Decide how to distribute the risky portion of the investment among the set of risky securities.

Deriving the Capital Market Line

- We can drive a formula for the CML algebraically. Based on the assumption of homogeneous expectations regarding the inputs in the portfolio construction process, all investors can create an efficient portfolio consisting of w_f placed in the risk-free asset and w_M in the market portfolio.

- Thus, $w_f + w_M = 1$. As the expected return of the portfolio, $E(R_p)$, is equal to the weighted average of the expected returns of the two assets, we have

$$E(R_p) = w_f R_f + w_M E(R_M)$$

Since we know that $w_f = 1 - w_M$, we can rewrite $E(R_p)$ as

$$E(R_p) = (1 - w_M)R_f + w_M E(R_M)$$

which can be simplified to

$$E(R_p) = R_f + w_M(E(R_M) - R_f)$$

Deriving the Capital Market Line

- Since the return of the risk-free asset and the return of the market portfolio are uncorrelated and the variance of the risk-free asset is equal to zero, the variance of the portfolio is given by

$$\sigma_p^2 = var(R_p) = w_f^2 var(R_f) + w_M^2 var(R_M) + 2w_f w_M cov(R_f, R_M)$$
$$= w_M^2 var(R_M) = w_M^2 \sigma_M^2$$

In other words, the variance of the portfolio is represented by the weighted variance of the market portfolio. We can write

$$w_M = \frac{\sigma_p}{\sigma_M}$$

If we substitute the preceding result and rearrange terms, we get the explicit expression for the CML

$$E(R_p) = R_f + \left[\frac{E(R_M) - R_f}{\sigma_M} \right] \sigma_p \leftarrow \text{risk premium}$$

- The numerator of the bracketed expression is the expected return from investing in the market beyond the risk-free return. It is a measure of the reward for holding the risky market portfolio rather than the risk-free asset.

- The slope of the CML, measures the reward per unit of market risk. Since the CML represents the return offered to compensate for a perceived level of risk, each point on the CML is a balanced market condition, or equilibrium.

- The slope of the CML determines the additional return needed to compensate for a unit change in risk, which is why it is also referred to as the *equilibrium market price of risk*.

$$E(R_p) = R_f + \text{Market price of risk} \times \text{Quality of risk}$$

Portfolio Constraints Commonly Used in Practice

- Institutional features and investment policy decisions often lead to more complicated constraints and portfolio management objectives than those present in the classical format.

  [Notations: $\mathbf{w}_0$ - the current portfolio weights; $\mathbf{w}$ - the targeted portfolio weights; $\mathbf{x} = w - w_0$ - the amount to be traded.

- **Linear and Quadratic Constraints** Will discuss some of the more commonly used ones.

- **Long-only Constraints** When short-selling is not allowed, we require that $\mathbf{w} \geq \mathbf{0}$. This is a frequently used constraint, as many funds and institutional investors are prohibited from selling stocks short.

- **Turnover Constraints** High portfolio turnover can result in large transaction costs that make portfolio re-balancing inefficient. The most common turnover constraints limit turnover on each individual asset

$$|x_i| \leq U_i \text{ or } \sum_{i \in I} |x_i| \leq U_{portfolio}$$

- **Holding Constraints** A well-diversified portfolio should not exhibit large concentrations in any specific assets, industries, sectors, or countries. Maximal holdings in an individual asset can be controlled by the constraint

$$L_i \leq w_i \leq U_i$$

where L_i and U_i are vectors representing the lower and upper bounds of the holdings of asset i. To constrain the exposure to a set I_i, we can have

$$L_i \leq \sum_{j \in I_i} w_j \leq U_i$$

- **Risk Factor Constraints** In practice, it is very common for portfolio managers to use factor models to control for different risk exposures to risk factors. Let us assume that security returns have a factor structure with K risk factors:

$$R_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} F_k + \epsilon_i$$

where $F_k, k = 1, ..., K$ are the K factors common to all the securities, β_{ik} is the sensitivity of the i-th security to the k-th factor, and ϵ_i is the noise for the i-th security. To limit exposure to the k-th risk factor, we impose

$$\sum_{i=1}^{N} \beta_{ik} w_i \le U_k$$

where U_k denotes maximum exposure allowed. To construct a portfolio that is neutral to the k-th risk factor

$$\sum_{i=1}^{N} \beta_{ik} w_i = 0$$

- **Benchmark Exposure and Tracking Error Constraints**
  Many portfolio managers are faced with the objective of
  managing their portfolio relative to a benchmark. A portfolio
  manager might choose to limit the deviations of the portfolio
  weights from the benchmark weights:

$$||w - w_b|| \leq M \text{ or } \sum_{j \in I_i}(w_j - w_{bj}) \leq M_i$$

  where w_b is the benchmark weights, and I_i is a specific
  industry.

- However, the most commonly used metric to measure the
  deviation from the benchmark is the *tracking error*:

$$TEV_p = var(R_p - R_b) = var(\mathbf{w}'\mathbf{R} - \mathbf{w_b}'\mathbf{R}) = (\mathbf{w} - \mathbf{w_b})'\mathbf{\Sigma}(\mathbf{w} - \mathbf{w_b})$$

  where Σ is the covariance matrix of the asset returns. To
  limit the tracking error, we will have

$$(\mathbf{w} - \mathbf{w_b})'\mathbf{\Sigma}(\mathbf{w} - \mathbf{w_b}) \leq \sigma_{TE}^2$$

- **General Linear and Quadratic Constraints** The constraints described in this section are all linear or quadratic, that is they can be cast either as

$$A_w w \leq d_w \text{ or } A_x x \leq d_x \text{ or } A_b(w - w_b) \leq d_b$$

or as

$$w'Q_w w \leq q_w$$
$$x'Q_x x \leq q_x$$
$$(w - w_b)'Q_b(w - w_b) \leq q_b$$

These types of constraints can be dealt with directly within the quadratic programming framework, and there are very efficient algorithms available that are capable of solving practical portfolio optimization problems with thousands of assets in a matter of seconds.

- **Combinatorial and Integer Constraints** The following binary decision variable is useful in describing some combinatorial and integer constraints:

$$\delta_i = \left\{ \begin{array}{ll} 1, & \text{if } w_i \neq 0 \\ 0, & \text{if } w_i = 0 \end{array} \right.$$

where w_i denotes the portfolio weight of the i-th asset.

- **Cardinality Constraints** A portfolio manager might want to restrict the number of assets allowed in a portfolio. The cardinality constraint takes the form

$$\sum_{i=1}^{N} \delta_i = K$$

where K is a positive integer significantly less than the number of assets in the investment universe, N.

- **Minimum Holding and Transaction Size Constraints** The classical mean-variance optimization problem often results in a few large and many small positions. In practice, due to transaction costs and other ticket charges, small holdings are undesirable. In order to eliminate small holdings, threshold constraints are often used

$$|w_i| \geq L_{w_i}\delta_i, i = 1, ..., N$$

where L_{w_i} is the smallest holding size allowed for asset i.

- This approach can be used to eliminate small trades, because the fixed costs related to trading each individual security. In practice, few portfolio managers go to the extent of including constraints of this type in their optimization framework. Instead, a standard mean-variance optimization problem is solved and then, in a post-optimization step, generated portfolio weights or trades that are smaller than a certain threshold are eliminated.

- **Round Lot Constraints** In reality, securities are transacted in
  multiples of a minimum transaction lots, or rounds (e.g. 100
  or 500 shares). In order to model transaction round lots
  explicitly in the optimization problem, portfolio weights can be
  represented as

$$w_i = z_i \cdot f_i, i = 1, ..., N$$

where f_i is a fraction of portfolio wealth and z_i is an
integer number of round lots.

- In applying round lot constraints, the budget constraint

$$\sum_{i=1}^{N} w_i = 1$$

may not be exactly satisfied.

- **Round Lot Constraints** To accommodate this situation, the budget constraint is relaxed with undershoot and overshoot variables, $\epsilon^- \geq 0$ and $\epsilon^+ \geq 0$, so that

$$\sum_{i=1}^{N} w_i + \epsilon^- - \epsilon^+ = 1$$

subject to

$$z'\boldsymbol{\Lambda}l + \epsilon^- - \epsilon^+ = \mathbf{1}, l' = [1, ..., 1]$$

where λ and γ are parameters chosen by the portfolio manager.

- Normally, the inclusion of round lot constraints to the mean-variance optimization problem only produces a small increase in risk for a pre-specified expected return.

  Furthermore, the portfolios obtained in this manner cannot be obtained by simply rounding the portfolio weights from a standard mean-variance optimization to the nearest round lot.

Portfolio Optimization with Other Risk Measures

- Many risks and undesirable scenarios faced by a portfolio
  manager cannot be captured solely by the variance of the
  portfolio. Consequently, especially in cases of significant
  non-normality, the classical mean-variance approach will not
  be a satisfactory portfolio allocation model.

- Since about the mid-1990s, considerable thought and
  innovation in the financial industry have been directed toward
  creating a management of risk and its measurement, and
  toward improving the management of risk in financial
  portfolios.

- The race for inventing the best risk measure for a given
  situation or portfolio is still ongoing, and the choice to some
  extent remains an art. We distinguish between two types of
  risk measures: (1) dispersion and (2) downside measures.