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The Capacitor

A capacitor is a two-terminal element that is a model of two conducting plates separated by a non-conducting material.
The charge deposited on the plates of a capacitor is proportional to the applied voltage:

\[ q = Cv \]

\( C \) is the capacitance, which has the units of farads (F); 1F = 1C/V.

For the capacitor shown, the capacitance is given by:

\[ C = \frac{\varepsilon A}{d} \]

where \( \varepsilon \) is the dielectric constant, \( A \) is the area of plates, and \( d \) is the space between plates.
When we first connect a battery to the capacitor, a current flows while the charges flow from one plate to the other.

Since
\[ i = \frac{dq}{dt} \]

we get:
\[ i = C \frac{dv}{dt} \]
Using the properties of superposition and homogeneity, we can show that the current-voltage relationship for the model of capacitor is a *linear* relation.

Circuit symbol for a capacitor:
The Capacitor

The voltage in terms of the current

\[ v = \frac{1}{C} \int_{-\infty}^{t} i \, d\tau \]

\[ = \frac{1}{C} \int_{-\infty}^{t_0} i \, d\tau + \frac{1}{C} \int_{t_0}^{t} i \, d\tau \]

\[ = v(t_0) + \frac{1}{C} \int_{t_0}^{t} i \, d\tau \]

Here, \( v(t_0) \) is called the initial condition.
Energy Storage in a Capacitor

- The capacitor stores energy by virtue of separation of charges between the capacitor plates.
- The forces acting on the charges stored in a capacitor result from an electric field.
- An electric field is defined as the force acting on a unit positive charge in a specified region.
- The energy required originally to separate the charges is now stored by the capacitor in the electric field.
The energy stored in a capacitor is

\[ w_c(t) = \int_{-\infty}^{t} v i d\tau \]

\[ = \int_{-\infty}^{t} vC \frac{dv}{d\tau} d\tau = C \int_{v(-\infty)}^{v(t)} v dv \]

\[ = \frac{1}{2} Cv^2 \bigg|_{v(-\infty)}^{v(t)} \]

\[ = \frac{1}{2} Cv(t)^2 \quad \text{if the capacitor was uncharged at } t = -\infty. \]
Since \( q = CV \),

\[
\omega_c = \frac{1}{2C} q(t)^2
\]

The voltage and charge on a capacitor cannot change instantaneously:

\[
V(0+) = V(0-)
\]
Parallel Capacitors

Using KCL, we have

\[ i = i_1 + i_2 + \cdots + i_N = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt} \]

\[ = \left( \sum_{n=1}^{N} C_n \right) \frac{dv}{dt} \]
So

\[ C_p = C_1 + C_2 + \cdots + C_N = \sum_{n=1}^{N} C_n \]
Series Capacitors

Using KVL, we get

\[ v = v_1 + v_2 + \cdots + v_N = \frac{1}{C_1} \int_{t_0}^{t} id\tau + v_1(t_0) + \cdots + \frac{1}{C_N} \int_{t_0}^{t} id\tau + v_N(t_0) \]

\[ = \left( \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \right) \int_{t_0}^{t} id\tau + \sum_{n=1}^{N} v_n(t_0) \]

At \( t = t_0 \), \( v(t_0) = \sum_{n=1}^{N} v_n(t_0) \),

So: \( v = \left( \sum_{n=1}^{N} \frac{1}{C_n} \right) \int_{t_0}^{t} id\tau + v(t_0) \)
\[
\frac{1}{C_s} = \sum_{n=1}^{N} \frac{1}{C_n}
\]
The Inductor

An inductor is a two-terminal element consisting of a winding of $N$ turns for introducing inductance into an electric circuit.

Inductor connected to a battery
Inductance is a measure of the ability of a device to store energy in the form of a magnetic field.

The unit of inductance is Henrys (H).
The passive sign convention for an inductor requires the current to flow into the positive terminal.

Voltage-current relationship:

\[ v = L \frac{di}{dt} \]
Inductors

- The current in an inductance cannot change instantaneously.
- Current-voltage relationship:

\[ i = \frac{1}{L} \int_{t_0}^{t} v(\tau) d\tau + i(t_0) \]
Energy Storage in an Inductor

\[ p = vi = \left( L \frac{di}{dt} \right) i \]

\[ w = \int_{t_0}^{t} pd\tau = L \int_{i(t_0)}^{i(t)} idi \]

\[ = \frac{L}{2} i^2(t) - \frac{L}{2} i^2(t_0) \]

If \( i(t_0) = 0 \), \( w = \frac{L}{2} i^2(t) \).
Series Inductors

Since \( v = v_1 + v_2 + \cdots + v_N \)  
\[
= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}
\]
\[
= \left( \sum_{n=1}^{N} L_n \right) \frac{di}{dt},
\]
So, \( L_s = \sum_{n=1}^{N} L_n \)
Parallel Inductors

Since $i = i_1 + i_2 + \cdots + i_N = \sum_{n=1}^{N} \left( \frac{1}{L_n} \int_{t_0}^{t} v(\tau) d\tau + i_n(t_0) \right)$

$$= \left( \sum_{n=1}^{N} \frac{1}{L_n} \right) \int_{t_0}^{t} v(\tau) d\tau + \sum_{n=1}^{N} i_n(t_0)$$

So, $\frac{1}{L_p} = \sum_{n=1}^{N} \frac{1}{L_n}$, $i(t_0) = \sum_{n=1}^{N} i_n(t_0)$
RL and RC Circuits

- RL and RC circuits are first-order circuits because their voltages and currents are described by first-order differential equations.
- We develop a general method that can be used to find the response of RL and RC circuits to an abrupt change in a dc voltage or current source.
RL and RC Circuits

No matter how complex the circuit, if it can be reduced to the Thevenin or Norton equivalent connected to the terminals of an equivalent inductor or capacitor, it is always a first-order circuit.
RC Circuit

- A first-order circuit containing a capacitor.
- After the switch closes, the circuit connected to the capacitor is replaced by its Thévenin equivalent circuit.
- We aim to determine the voltage after the switch closes.
RL Circuit

- A first-order circuit containing an inductor
- After the switch closes, the circuit connected to the inductor is replaced by its Norton equivalent circuit
- We aim to determine the current after the switch closes.
First-Order Circuits Response

We can derive a general form of first-order differential equation from the above two circuits:

$$\frac{d}{dt} x(t) + \frac{x(t)}{\tau} = K$$

where $\tau$ is called the time constant.
Solving this first-order differential equation gives

\[ x(t) = K\tau + Ae^{\frac{t}{\tau}} \]

where \( A = x(0) - K\tau; \)

\[ \tau = \frac{x(\infty) - x(0)}{\frac{dx(t)}{dt} \bigg|_{t=0}} \]
Response of RC and RL Circuits

**RC circuit:**

\[ v(t) = V_{oc} + (v(0) - V_{oc}) e^{-\frac{t}{R_{TH}C}} \]

**RL circuit:**

\[ i(t) = I_{sc} + (i(0) - I_{sc}) e^{-\frac{R_{TH}}{L}t} \]
Procedure for analyzing first-order circuits:

- Reduce the circuit to an proper equivalent form using Thevenin and Norton equivalent circuits.
- Identify the variable of interest for the circuit. For RC circuits, it is usually the voltage; and for RL circuit, it is usually the current.
- Determine the initial value of the variable at t=0. Note that the capacitor voltage and inductor current have the same values at t(0+) and t(0-).
- Calculate the time constant.
- Use the solution equations for RC and RL circuits.
- Once you know the variable for inductor and capacitor, calculate the relevant values for all other components.
Second-Order Circuits

A series RLC circuit:
Response of Second-Order Circuits

Transient response is described by the roots of the network’s characteristic equation:

\[ \frac{d^2 x}{dt^2} + \frac{1}{RC} \frac{dx}{dt} + \frac{1}{LC} x = 0 \]

CE: \( s^2 + \frac{1}{RC} s + \frac{1}{LC} s = 0 \)

\[ s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]
Response of Second-Order Circuits

- Two real roots, \( s = -r_1, -r_2 \)
  overdamped
Response of Second-Order Circuits

- Two equal roots, \( s = -r_1, -r_1 \)
critically damped
Complex roots, \( s = -r_1 \pm j \omega_d \)

underdamped