Final Exam

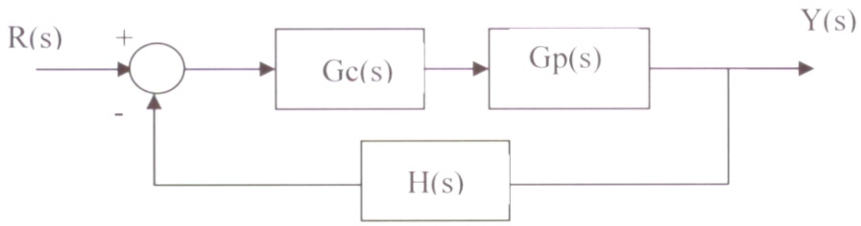
EE 575 Introduction to Control Theory (Spring 2009)

Name: ____________________

SID: ____________________

Do all work in the spaces provided, and you can use the back of the paper if needed. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) The block diagram of a linear control system is shown:



where $G_p(s) = \dfrac{1}{s(s+7)}$, $H(s) = 1$.

We'd like to design a compensator $G_c(s) = \dfrac{K(s+a)}{s+b}$ to put the dominate closed-loop poles at $-7 \pm j13.66$.

1) Choose a=3, find the location of the compensator pole (i.e., the value of -b).
2) Find the gain K.
3) What is the steady-state error for a unit step input?

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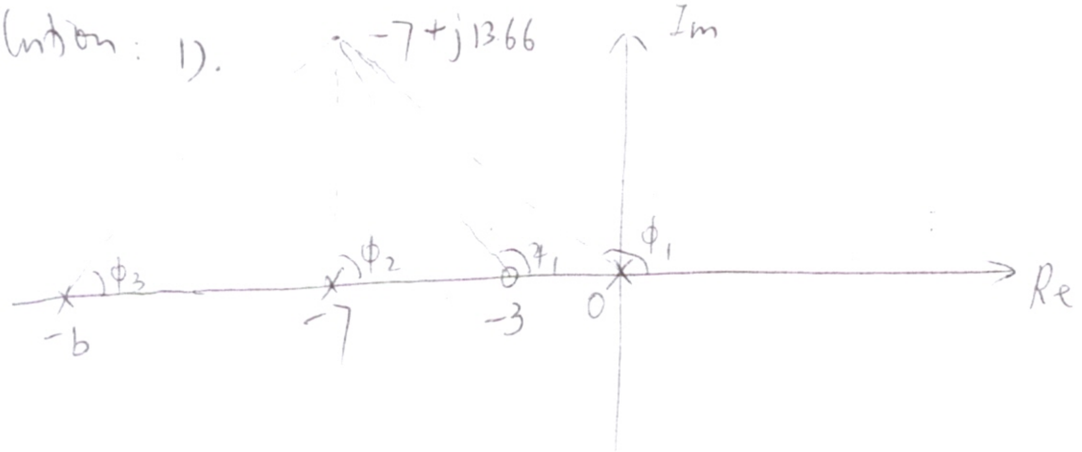
Solution: 1).



$$G_c G_p(s) = \frac{k(s+a)}{s(s+7)(s+b)}$$

Use angle condition, $z_1 - (\phi_1 + \phi_2 + \phi_3) = -180^\circ$

$$z_1 = 180^\circ - \tan^{-1}\frac{13.66}{4} = 106.3$$

$$\phi_1 = 180^\circ - \tan^{-1}\frac{13.66}{7} = 117.1$$

$$\phi_2 = 90^\circ$$

$$\Rightarrow \phi_3 = 180^\circ + \psi_1 - \phi_1 - \phi_2 = 79.2$$

Since $\tan\phi_3 = \frac{13.66}{b-7}$

$$\Rightarrow b = \frac{13.66}{\tan 79.2} + 7 = 9.6$$

2). Use magnitude condition:

$$\left| \frac{k(s+3)}{s(s+7)(s+9.6)} \right| \Bigg|_{s=-7+j13.66} = 1$$

$$\Rightarrow k = \frac{|s| \cdot |s+7| \cdot |s+9.6|}{|s+3|} \Bigg|_{s=-7+j13.66}$$

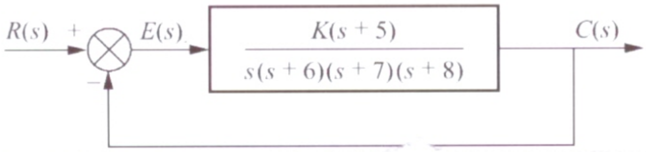
$$= \frac{15.35 \times 13.66 \times 13.91}{14.23}$$

$$\simeq 205$$

3). $k_p = \lim\limits_{s \to 0} G_c G_p = \infty$

$$e_{ss} = \frac{1}{1+k_p} = 0.$$

Problem 2. (25 points) Given the control system shown below, find the value of K so that there is a 10% steady state error for a unit ramp input.



Solution:

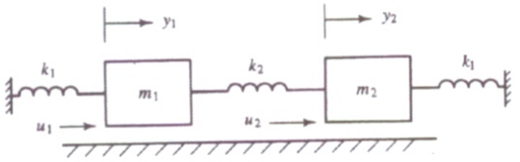
$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{K(s+5)}{(s+6)(s+7)(s+8)}$$

$$= \frac{5K}{6\times 7\times 8} = \frac{5K}{336}$$

$$e_{ss} = \frac{1}{K_v} = \frac{336}{5K} = 10\%$$

$$\Rightarrow \quad K = 672.$$

Problem 3. (25 points) Consider a mechanical system shown in the figure:



Its differential equation is given by

$$m_1 \frac{d^2 y_1(t)}{dt^2} + (k_1 + k_2) y_1(t) - k_2 y_2 = u_1$$

$$m_2 \frac{d^2 y_2(t)}{dt^2} - k_2 y_1(t) + (k_1 + k_2) y_2 = u_2$$

Express the system in the state space representation, considering $y_1(t)$ and $y_2(t)$ to be the output and $u_1(t)$ and $u_2(t)$ to be the input (note that this is a two-input two-output system).

Solution: Let $X = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = -\frac{k_1+k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2 + \frac{u_1}{m_1}$$

$$\dot{X}_3 = X_4, \quad \dot{X}_4 = \frac{k_2}{m_2} y_1 - \frac{k_1+k_2}{m_2} y_2 + \frac{u_2}{m_2}$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_1+k_2}{m_2} & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X$$

Problem 4. (25 points) Given the state space representation of the plant,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 50 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

design a state feedback controller such that the closed-loop characteristic equation has a damping ratio $\xi = 0.707$, and undamped natural frequency $\omega_n = 10$.

Solution: Desired closed-loop CE:

$$s^2 + 2\xi w_n s + w_n^2 = 0$$

$$s^2 + 14.14s + 100 = 0$$

Design state feedback $u = -kx = -[k_1 \ \ k_2]x$

Closed-loop CE:

$$\det[sI - (A - BK)] = 0$$

$$\det\left[sI - \begin{bmatrix} -1 & 1 \\ 2-k_1 & -2-k_2 \end{bmatrix} \right] = 0$$

$$\Rightarrow s^2 + (3+k_2)s + (k_1+k_2) = 0$$

$$\Rightarrow \begin{cases} 3 + k_2 = 14.14 \\ k_1 + k_2 = 100 \end{cases} \Rightarrow \begin{cases} k_1 = 88.86 \\ k_2 = 11.14 \end{cases}$$

So $k = [88.86 \quad 11.14]$, and $u = -kx$.