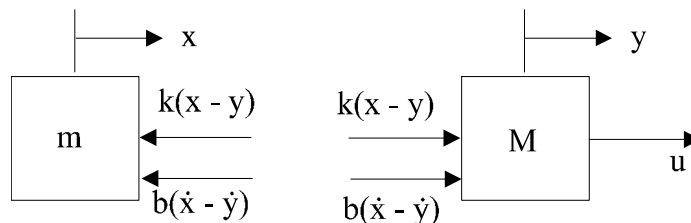


8. In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Figure 2.6 where there is flexibility of the solar panels. Figure 2.36 depicts such a situation, where a force  $u$  is applied to the mass  $M$  and another mass  $m$  is connected to it. The coupling between the objects is often modeled by a spring constant  $k$  with a damping coefficient  $b$ , although the actual situation is usually much more complicated than this.

- Write the equations of motion governing this system.
- Find the transfer function between the control input,  $u$ , and the output,  $y$ .

**Solution:**

- The FBD for the system is



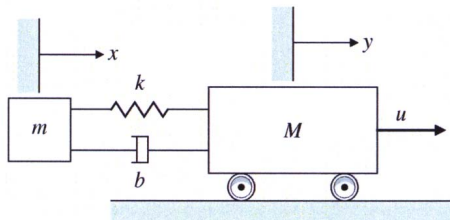
which results in the equations

$$\begin{aligned} m\ddot{x} &= -k(x - y) - b(\dot{x} - \dot{y}) \\ M\ddot{y} &= u + k(x - y) + b(\dot{x} - \dot{y}) \end{aligned}$$

or

$$\begin{aligned} \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\ -\frac{k}{M}x - \frac{b}{M}\dot{x} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} &= \frac{1}{M}u \end{aligned}$$

Figure 2.36: Schematic of a system with flexibility



(b) If we make Laplace Transform of the equations of motion

$$\begin{aligned} s^2 X + \frac{k}{m} X + \frac{b}{m} s X - \frac{k}{m} Y - \frac{b}{m} s Y &= 0 \\ -\frac{k}{M} X - \frac{b}{M} s X + s^2 Y + \frac{k}{M} Y + \frac{b}{M} s Y &= \frac{1}{M} U \end{aligned}$$

In matrix form,

$$\begin{bmatrix} ms^2 + bs + k & -(bs + k) \\ -(bs + k) & Ms^2 + bs + k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

From Cramer's Rule,

$$\begin{aligned} Y &= \frac{\det \begin{bmatrix} ms^2 + bs + k & 0 \\ -(bs + k) & U \end{bmatrix}}{\det \begin{bmatrix} ms^2 + bs + k & -(bs + k) \\ -(bs + k) & Ms^2 + bs + k \end{bmatrix}} \\ &= \frac{ms^2 + bs + k}{(ms^2 + bs + k)(Ms^2 + bs + k) - (bs + k)^2} U \end{aligned}$$

Finally,

$$\begin{aligned} \frac{Y}{U} &= \frac{ms^2 + bs + k}{(ms^2 + bs + k)(Ms^2 + bs + k) - (bs + k)^2} \\ &= \frac{ms^2 + bs + k}{mMs^4 + (m + M)bs^3 + (M + m)ks^2} \end{aligned}$$

16. For a second-order system with transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3},$$

determine the following:

- (a) The DC gain;
- (b) The final value to a step input.

**Solution:**

(a) DC gain  $G(0) = \frac{3}{-3} = -1$

(b)  $\lim_{t \rightarrow \infty} y(t) = ?$

$$s^2 + 2s + 3 = 0 \implies s = 1, -3$$

Since the system has an unstable pole, the Final Value Theorem is not applicable. The output is unbounded.

38. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

$$(a) \quad KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$$

$$(b) \quad KG(s) = \frac{2(s+4)}{s^2(s+1)}$$

$$(c) \quad KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$$

**Solution:**

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0$$

$$\begin{array}{rcl} s^4 & : & 1 \quad 3 \quad 8 \\ s^3 & : & 2 \quad 8 \\ s^2 & : & a \quad b \\ s^1 & : & c \\ s^0 & : & d \end{array}$$

where

$$a = \frac{2 \times 3 - 8 \times 1}{2} = -1 \quad b = \frac{2 \times 8 - 1 \times 0}{2} = 8$$

$$c = \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24$$

$$d = b = 8$$

2 sign changes in first column  $\implies$  2 roots not in LHP  $\implies$  unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0$$

The Routh's array is,

$$\begin{array}{rcl} s^3 & : & 1 \quad 2 \\ s^2 & : & 1 \quad 8 \\ s^1 & : & -6 \\ s^0 & : & 8 \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0$$

$$\begin{array}{rcll} s^5 & : & 1 & 3 & 4 \\ s^4 & : & 2 & 7 & 4 \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & b_2 & \\ s^1 & : & c_1 & & \\ s^0 & : & d_1 & & \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{6-7}{2} = \frac{-1}{2} & a_2 &= \frac{8-4}{2} = 2 \\ b_1 &= \frac{-7/2-4}{-1/2} = 15 & b_2 &= \frac{-4/2-0}{-1/2} = 4 \\ c_1 &= \frac{30+2}{15} = \frac{32}{15} \\ d_1 &= 4 \end{aligned}$$

2 sign changes in the first column  $\implies$  2 roots not in the LHP  $\implies$  unstable.

39. Use Routh's stability criterion to determine how many roots with positive real parts the following equations have:

(a)  $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$ .

(b)  $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$ .

(c)  $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$ .

(d)  $s^3 + s^2 + 20s + 78 = 0$ .

(e)  $s^4 + 6s^2 + 25 = 0$ .

**Solution:**

(a)

$$s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$$

$s^4$	:	1	32	100
$s^3$	:	8	80	
$s^2$	:	22	100	
$s^1$	:	$80 - \frac{800}{22} = 43.6$		
$s^0$	:	100		

$\implies$  No roots not in the LHP

(b)

$$s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$$

$s^5$	:	1	30	344
$s^4$	:	10	80	480
$s^3$	:	22	296	
$s^2$	:	-545	480	
$s^1$	:	490		
$s^0$	:	480		

$\implies$  2 roots not in the LHP.

(c)

$$s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$$

There are roots in the RHP (not all coefficients are  $>0$ ).

$$\begin{array}{rcl}
 s^4 & : & 1 \quad 7 \quad 8 \\
 s^3 & : & 2 \quad -2 \\
 s^2 & : & 8 \quad 8 \\
 s^1 & : & -4 \\
 s^0 & : & 8
 \end{array}$$

$\implies$  2 roots not in the LHP.

(d) The Routh array is,

$$\begin{array}{rcl}
 s^3 & : & 1 \quad 20 \\
 s^2 & : & 1 \quad 78 \\
 s^1 & : & -58 \\
 s^0 & : & 78
 \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(e)

$$a(s) = s^4 + 6s^2 + 25 = 0$$

Two coefficients are missing so there are roots outside the LHP.

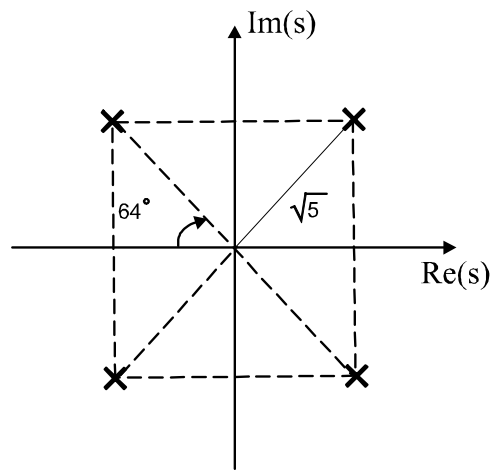
Create a new row by  $\frac{da(s)}{ds}$

$$\begin{array}{rcl}
 s^4 & : & 1 \quad 6 \quad 25 \\
 s^3 & : & 4 \quad 12 \quad \longleftarrow \text{new row} \\
 s^2 & : & 3 \quad 25 \\
 s^1 & : & 12 - \frac{100}{3} = -21.3 \\
 s^0 & : & 25
 \end{array}$$

$\implies$  2 roots not in the LHP

check:

$$\begin{aligned}
 a(s) &= 0 \implies s^2 = -3 \pm 4j = 5e^{j(\pi \pm 0.92)} \\
 s &= \sqrt{5}e^{j(\frac{\pi}{2} \pm 0.46) + n\pi j} \quad n = 0, 1
 \end{aligned}$$

Problem 3.39:  $s$ -plane pole locations.

40. Find the range of  $K$  for which all the roots of the following polynomial are in the LHP:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the  $s$ -plane for various values of  $K$ .

**Solution:**

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

$$\begin{array}{rcl} s^5 & : & 1 \quad 10 \quad 5 \\ s^4 & : & 5 \quad 10 \quad K \\ s^3 & : & a_1 \quad a_2 \\ s^2 & : & b_1 \quad K \\ s^1 & : & c_1 \\ s^0 & : & K \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{5(10) - 1(10)}{5} = 8 & a_2 &= \frac{5(5) - 1(K)}{5} = \frac{25 - K}{8} \\ b_1 &= \frac{(a_1)(10) - (5)(a_2)}{a_1} = \frac{55 + K}{8} \\ c_1 &= \frac{(b_1)(a_2) - (a_1)(K)}{b_1} = \frac{-(K^2 + 350K - 1375)}{5(55 + K)} \end{aligned}$$

For stability: all terms in first column  $> 0$

$$\begin{aligned} (1) \quad b_1 &= \frac{55 + K}{8} > 0 \implies K > -55 \\ (2) \quad c_1 &= \frac{-(K^2 + 350K - 1375)}{5(55 + K)} > 0, \frac{-(K - 3.88)(K + 354)}{5(55 + K)} > 0 \implies -55 < K < 3.88 \\ (3) \quad d_1 &= K > 0 \end{aligned}$$

Combining (1), (2), and (3)  $\implies 0 < K < 3.88$ . If we plot the roots of the polynomial for various values of  $K$  we obtain the following root locus plot (see Chapter 5),

