Chapter 7

State-Space Design

Problems and Solutions for Section 7.3: Block diagrams and State Space

1. A schematic for the satellite and scientific probe for the Gravity Probe-B (GP-B) experiment that was launched on April 30, 2004 is sketched in Fig. 7.82. Assume that the mass of the spacecraft plus helium tank, m_1, is 2000 kg and the mass of the probe, m_2, is 1000 kg. A rotor will float inside the probe and will be forced to follow the probe with a capacitive forcing mechanism. The spring constant of the coupling, k, is 3.2×10^6. The viscous damping b is 4.6×10^3.

   (a) Write the equations of motion for the system consisting of masses m_1 and m_2 using the inertial position variables, y_1 and y_2.

   (b) The actual disturbance u is a micrometeorite, and the resulting motion is very small Therefore, rewrite your equations with the scaled variables $z_1 = 10^6 y_1$, $z_2 = 10^6 y_2$, and $v = 1000u$.

   (c) Put the equations in state-variable form using the state $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$, the output $y = z_2$, and the input an impulse, $u = 10^{-3}\delta(t)$ N$\cdot$ sec on mass m_1.

   (d) Using the numerical values, enter the equations of motion into MATLAB in the form

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}v \qquad (1)$$

$$y = \mathbf{H}\mathbf{x} + Jv \qquad (2)$$

   and define the MATLAB system: sysGPB = ss(F,G,H,J). Plot the response of y caused by the impulse with the MATLAB command impulse(sysGPB). This is the signal the rotor must follow.

   (e) Use the MATLAB commands p = eig(F) to find the poles (or roots) of the system and z = tzero(F,G,H, J) to find the zeros of the system.

   Solution:

   (a) The rotor is not part of the problem and can be ignored in writing the equations of motion

$$\begin{aligned} m_1\ddot{y}_1 &= u - k(y_1 - y_2) - b(\dot{y}_1 - \dot{y}_2) \\ m_2\ddot{y}_2 &= -k(y_2 - y_1) - b(\dot{y}_2 - \dot{y}_1) \end{aligned}$$

(b) Let's put in the values for the parameters as well as scale the variables as requested.

$$\begin{aligned}
2000\left(10^{-6}\ddot{z}_1\right) &= \frac{1}{1000}v - 10^{-6}(3.2\times 10^6)\,(z_1 - z_2) - 10^{-6}(4.6\times 10^3)\,(\dot{z}_1 - \dot{z}_2) \\
1000\left(10^{-6}\ddot{z}_2\right) &= -10^{-6}(3.2\times 10^6)\,(z_2 - z_1) - 10^{-6}(4.6\times 10^3)\,(\dot{z}_2 - \dot{z}_1)
\end{aligned}$$

which becomes

$$\begin{aligned}
\ddot{z}_1 &= -(1.6\times 10^3)\,(z_1 - z_2) - (2.3)\,(\dot{z}_1 - \dot{z}_2) + \frac{1}{2}v \\
\ddot{z}_2 &= -(3.2\times 10^3)\,(z_2 - z_1) - (4.6)\,(\dot{z}_2 - \dot{z}_1)
\end{aligned}$$

(c) The state-variable form for $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$ is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1.6\times 10^3)\,(x_1 - x_3) - (2.3)\,(x_2 - x_4) + \frac{1}{2}v \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -(3.2\times 10^3)\,(x_3 - x_1) - (4.6)\,(x_4 - x_2)
\end{aligned}$$

or, in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.6\times 10^3 & -2.3 & 1.6\times 10^3 & 2.3 \\ 0 & 0 & 0 & 1 \\ 3.2\times 10^3 & 4.6 & -3.2\times 10^3 & -4.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} v$$

and the output equation is

$$y = [0 \quad 0 \;\; 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0$$

(d) The system matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.6\times 10^3 & -2.3 & 1.6\times 10^3 & 2.3 \\ 0 & 0 & 0 & 1 \\ 3.2\times 10^3 & 4.6 & -3.2\times 10^3 & -4.6 \end{bmatrix}$$

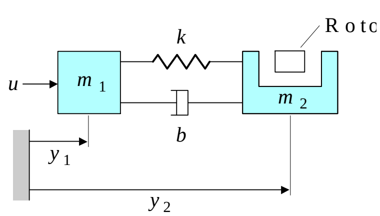


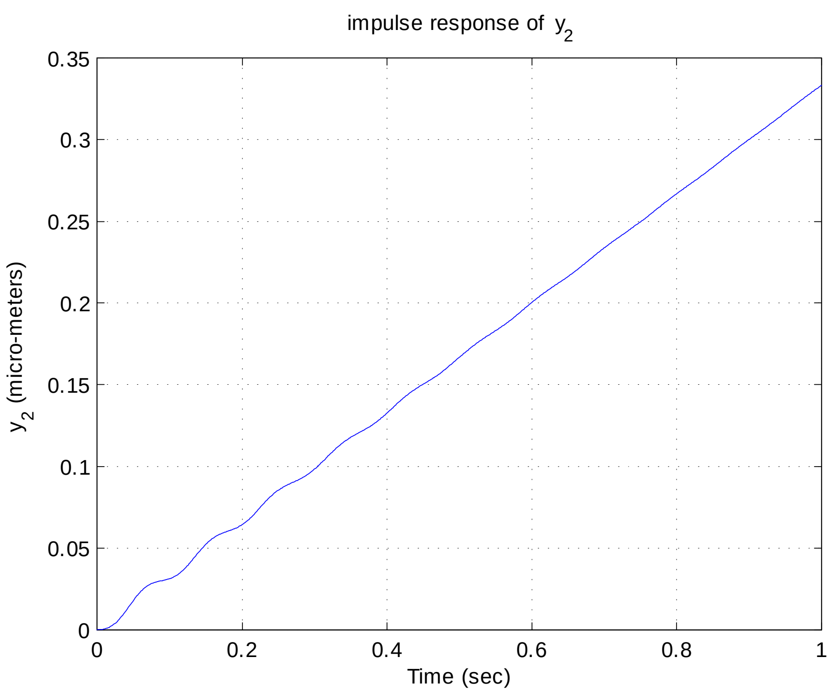
Figure 7.82: [Text Fig. 7.82] Schematic diagram of the GP-B satellite and probe.

$$\mathbf{G} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = [0 \quad 0 \quad 1 \quad 0] \quad \text{and} \quad J = 0$$

plus the MATLAB statements:

```
sysGPB = ss(F,G,H,J);
t=0:0.001:1;
y=impulse(sysGPB,t);    % u = 10^-3 implies that v=1
plot(t,y)
```

produce the plot below.



Impulse response for Problem 7.1.

The micro meteorite hits the first mass and imparts a velocity of 0.33μm/sec to the two mass system. It also excites the resonant mode of relative motion between the masses that dies out in less than a second.

Problems and Solutions for Section 7.4: Analysis of the State Equations

2. Give the state description matrices in control-canonical form for the following transfer functions:

(a) $\dfrac{1}{4s+1}$

(b) $\dfrac{5(s/2+1)}{(s/10+1)}$

(c) $\dfrac{2s+1}{s^2+3s+2}$

(d) $\dfrac{s+3}{s(s^2+2s+2)}$

(e) $\dfrac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)}$

Solution:

(a) $F = -0.25,\ G = 1,\ H = 0.25,\ J = 0.$

(b) $F = -10,\ G = 1,\ H = -200,\ J = 25.$

Hint: Do a partial fraction expansion to find the J term first.

(c)

$$\mathbf{F} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 2 & 1 \end{bmatrix},\ J = [0].$$

(d)

$$\mathbf{F} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix},\ J = [0].$$

(e)

$$\mathbf{F} = \begin{bmatrix} -4 & -39 & -108 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 0 & 1 & 11 & 35 & 250 \end{bmatrix},\ J = [0].$$

3. Use the MATLAB function tf2ss to obtain the state matrices called for Problem 7.2.

Solution:

In all cases, simply form *num* and *den* given below and then use the MATLAB command [F,G,H,J] = tf2ss(num,den).

(a) $num = \left[\begin{array}{cc} 0 & 1 \end{array} \right]$, $den = \left[\begin{array}{cc} 4 & 1 \end{array} \right]$.

(b) $num = \left[\begin{array}{cc} 5/2 & 5 \end{array} \right]$, $den = \left[\begin{array}{cc} 1/10 & 1 \end{array} \right]$.

(c) $num = \left[\begin{array}{ccc} 0 & 2 & 1 \end{array} \right]$, $den = \left[\begin{array}{ccc} 1 & 3 & 2 \end{array} \right]$.

(d) $num = \left[\begin{array}{cccc} 0 & 0 & 1 & 3 \end{array} \right]$, $den = \left[\begin{array}{cccc} 1 & 2 & 2 & 0 \end{array} \right]$.

(e) $num = \left[\begin{array}{cccccc} 0 & 0 & 1 & 11 & 35 & 250 \end{array} \right]$, $den = \left[\begin{array}{ccccccc} 1 & 4 & 1 & 39 & 108 & 0 & 0 \end{array} \right]$.

Note that the answers are the same as for Problem 7.2.

Hint: The MATLAB function conv will save time when forming the numerator and denominator for part (e).

17. For each of the listed transfer functions, write the state equations in both control and observer canonical form. In each case draw a block diagram and give the appropriate expressions for **F**, **G**, and **H**.

a) $\frac{s^2-2}{s^2(s^2-1)}$ (control of an inverted pendulum by a force on the cart)

b) $\frac{3s+4}{s^2+2s+2}$

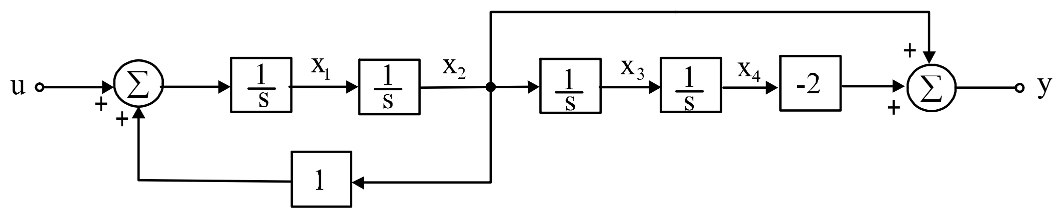
Solution:

(a)

$$\frac{Y(s)}{U(s)} = \frac{s^2-2}{s^2(s^2-1)}.$$

This transfer function can be realized in controller canonical form as shown below. From the figure, we have,

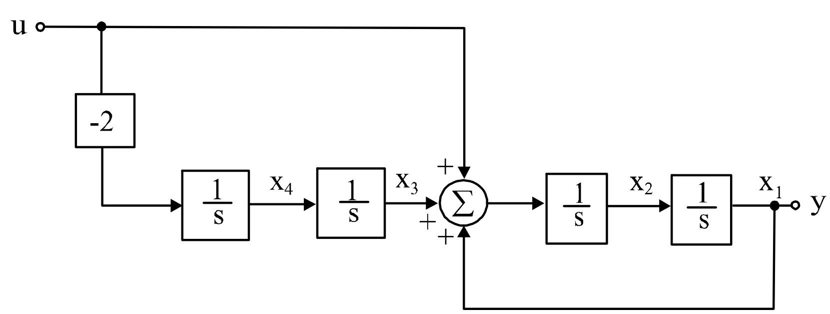
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix} \mathbf{x}.$$



Controller canonical form for the transfer function of Problem 7.17(a).

The block diagram for observer canonical form is shown below. From the figure, we have,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$



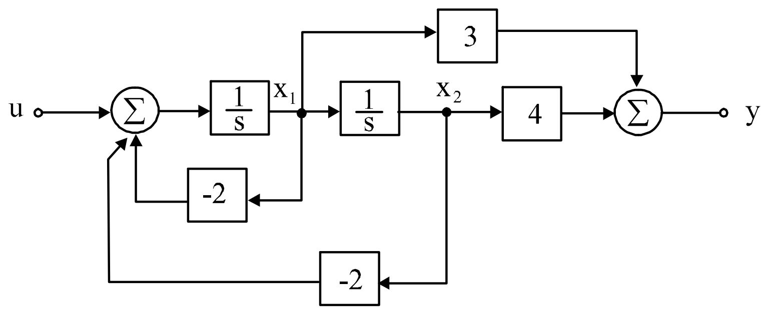
Observer canonical form for the transfer function of Problem 7.17(a).

(b)

$$\frac{Y(s)}{U(s)} = \frac{3s+4}{s^2+2s+2}.$$

This transfer function can be realized in Controller canonical form as shown below. From the figure, we have,

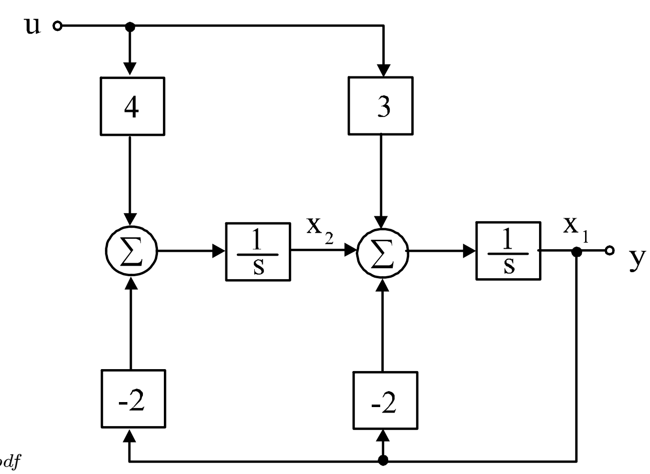
$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 3 & 4 \end{bmatrix} \mathbf{x}. \end{aligned}$$



Controller canonical form for the transfer function of Problem 7.17(b).

The block diagram for observer canonical form is shown below. From the figure, we have:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$



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Observer canonical form for the transfer function of Problem 7.17(b).