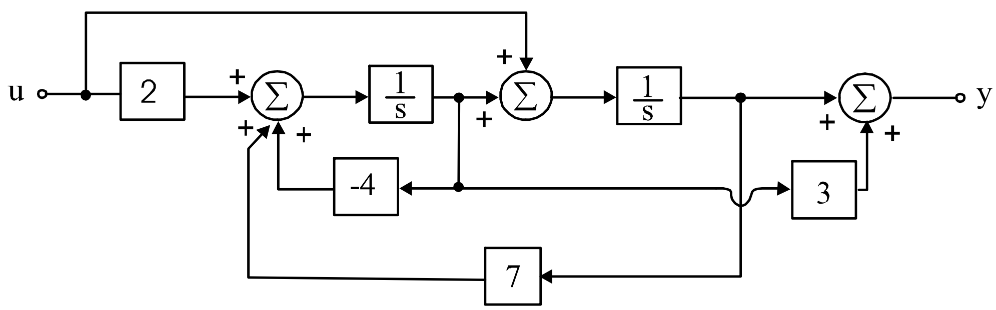
Problems and Solutions for Section 7.5: Control-Law Design for Full-State Feedback

19. Consider the plant described by,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x}.$$

a) Draw a block diagram for the plant with one integrator for each state variable.
b) Find the transfer function using matrix algebra.
c) Find the closed-loop characteristic equation if the feedback is
(1) $u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\mathbf{x}$; (2) $u = -Ky$.

Solution:



State realization showing integrators explicitly.

(a) See figure.

(b) Using the formula $G(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$, we obtain,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{7s + 27}{s^2 + 4s - 7}.$$

The MATLAB command ss2tf can also be used.

(c)

(i) State feedback, $u = -[K_1 \ K_2]\mathbf{x}$.

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{F} + \mathbf{GK}) &= \det \begin{bmatrix} \lambda + K_1 & -1 + K_2 \\ -7 + 2K_2 & \lambda + 4 + 2K_2 \end{bmatrix} \\ &= \lambda^2 + \lambda(4 + 2K_2 + K_1) + (6K_1 + 7K_2 - 7) = 0. \end{aligned}$$

(ii) Output feedback,

$$u = -Ky = -K \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x} = -\begin{bmatrix} K & 3K \end{bmatrix} \mathbf{x}.$$

This yields the following closed-loop characteristic equation:

$$\lambda^2 + \lambda(7K + 4) + (27K - 7) = 0.$$

Hints: If you have already solved the case for state feedback, simply plug $K_1 = K$ and $K_2 = 3K$ into the characteristic equation for state feedback and find the characteristic equation for output feedback. The output vector $\mathbf{H}$ fixes the ratio among the state variables. Secondly, although there were products of K_1 and K_2 when we were forming the determinant, they should all cancel in your final answer. The reason for this is that the characteristic equation $\dot{\mathbf{x}} = (\mathbf{F} - \mathbf{G}\mathbf{K})\mathbf{x}$ is *linear* in $\mathbf{K}$.

20. For the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x},$$

design a state feedback controller that satisfies the following specifications:

- Closed-loop poles have a damping coefficient $\zeta = 0.707$.

- Step-response peak time is under 3.14 sec.

Verify your design with MATLAB.

Solution:

For a second-order system, the specification on rise time can be translated into a value of $\omega_n$by the equation $\omega_d = \pi$. Then determine ω_n from $\omega_d = \omega_n\sqrt{1-\zeta^2}$. This yields $\omega_n = 1.414$. Using full state feedback, we would like the a characteristic equation to be,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 2 = 0.$$

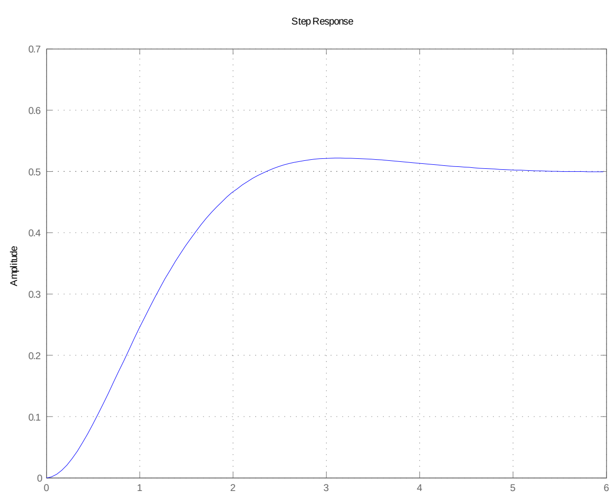
Using state feedback $u = -\mathbf{K}\mathbf{x}$, we get,

$$\dot{\mathbf{x}} = (\mathbf{F} - \mathbf{G}\mathbf{K})\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -6-k_1 & -5-k_2 \end{bmatrix}\mathbf{x}.$$

Hence the closed-loop characteristic equation is,

$$s^2 + (5+k_2)s + (6+k_1) = 0.$$

Comparing coefficients, $k_1 = -4$ and $k_2 = -3$. The MATLAB command `place` can also be used. The reference step can be simulated in MATLAB with $u = -\mathbf{K}\mathbf{x} + r$, and the MATLAB command `step`, as shown below.



Step response for Problem 7.20.

21. a) Design a state feedback controller for the following system so that the closed-loop step response has an overshoot of less than 25% and a 1% settling time under 0.115 sec.:

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}.\end{aligned}$$

b) Use the step command in MATLAB to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

Solution:

(a) For the overshoot specification,

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 25\% \Longrightarrow \zeta \cong 0.4.$$

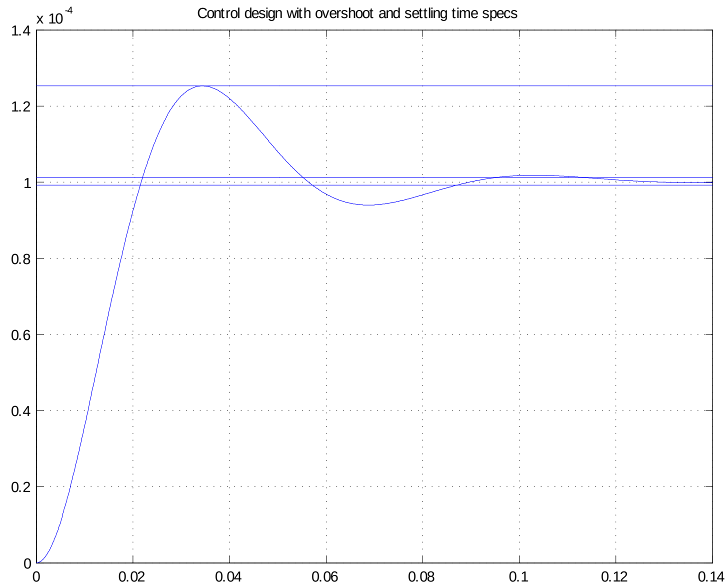
For the 1% settling time specification, we use,

$$e^{-\zeta\omega_n t_s} = 0.01 \Longrightarrow \omega_n = \frac{4.6}{\zeta t_s}.$$

(b) This can be implemented in MATLAB with the following code:

```
F = [0,1;0,-10];

G = [0;1];

H = [1,0];

J = 0;

zeta = 0.404; % Tweak values slightly so that specs are met.

ts = 0.114;

wn = 4.6/(ts*zeta);

p = roots([1, 2*zeta*wn, wn^2]);

k = place(F,G,p);

sysCL=ss(F-G*k,G,H,J)

step(sysCL);
```

The step response is shown on top of the next page.



Step response for Problem 7.21.