Midterm Exam

EE 602 Analytic Methods (Spring 2007)

Name: Solutions

SID: ____________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (20 points) Find the LU decomposition of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Solution: Exchange row 1 and row 2,

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \triangleq A_1$$

Elementary row operations,

$$E_2 E_1 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \triangleq U$$

$$PA = (E_2 E_1)^{-1} U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \triangleq LU$$

Problem 2. (20 points) Write the complete solution as a general homogeneous solution plus one particular solution.

$$x+3y+3z=1$$
$$2x+6y+9z=5$$
$$-x-3y+3z=5$$

Solution: Express the equations in the form of $AX=b$,

$$A=\begin{bmatrix}1&3&3\\2&6&9\\-1&-3&3\end{bmatrix}, \quad b=\begin{bmatrix}1\\5\\5\end{bmatrix}$$

Perform row operation

$$E_1A=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&3&3\\2&6&9\\-1&-3&3\end{bmatrix}=\begin{bmatrix}1&3&3\\0&0&3\\-1&-3&3\end{bmatrix}\triangleq A_1$$

$$E_2A_1=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\cdot\begin{bmatrix}1&3&3\\0&0&3\\-1&-3&3\end{bmatrix}=\begin{bmatrix}1&3&3\\0&0&3\\0&0&6\end{bmatrix}\triangleq A_2$$

$$E_3A_2=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\cdot\begin{bmatrix}1&3&3\\0&0&3\\0&0&6\end{bmatrix}=\begin{bmatrix}1&3&3\\0&0&3\\0&0&0\end{bmatrix}$$

So $E_3E_2E_1A=\begin{bmatrix}1&3&3\\0&0&3\\0&0&0\end{bmatrix}\triangleq A_3$

Homogeneous solution: $A_3X=0 \Rightarrow$ A basis for all solutions is $\begin{bmatrix}-3\\1\\0\end{bmatrix}$

All possible solutions are multiples of the basis.

Particular solution: $E_3E_2E_1AX=E_3E_2E_1b=\begin{bmatrix}1\\3\\0\end{bmatrix}$.

one particular solution is $\begin{bmatrix}-2\\0\\1\end{bmatrix}$

So the complete solution is $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\0\\1\end{bmatrix}+\begin{bmatrix}-3\\1\\0\end{bmatrix}\cdot k$ for any k.

Problem 3. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 9 \end{bmatrix}$$

a) **How many linearly independent column vectors does A have?**
b) **What is the rank of A?**

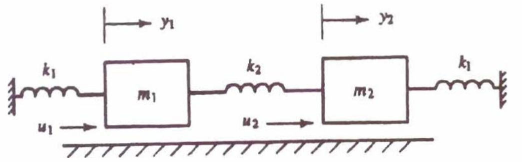
Solution: Perform elementary row operations.

$$A \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 2 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that $\text{rank}(A) = 2$,

there are 2 linearly independent column vectors

Problem 4. (20 points) Consider a mechanical system shown in the figure:



Its differential equation is given by

$$m_1 \frac{d^2 y_1(t)}{dt^2} + (k_1 + k_2) y_1(t) - k_2 y_2 = u_1$$

$$m_2 \frac{d^2 y_2(t)}{dt^2} - k_2 y_1(t) + (k_1 + k_2) y_2 = u_2$$

Express the system in the state space representation, considering $y_1(t)$ and $y_2(t)$ to be the output and $u_1(t)$ and $u_2(t)$ to be the input (note that this is a two-input two-output system).

Solution: Let $X = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k_1+k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2 + \frac{u_1}{m_1}$$

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = \frac{k_2}{m_2} y_1 - \frac{k_1+k_2}{m_2} y_2 + \frac{u_2}{m_2}$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_1+k_2}{m_2} & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X.$$

Problem 5. (20 points) Suppose using the state variable x we have

$$\dot{x} = Ax + Bu \quad \cdots ①$$

$$y = Cx + Du \quad \cdots ②$$

Consider a change of variables x = Mz where M is non-singular.

a) **What are the new state equations?**

b) **Show that the transfer function did not change.**

Solution: a). $x = Mz$, M is non-singular

$\Rightarrow z = M^{-1}x$

To get the new state equation, multiply M^{-1} from the left to the original state equation,

$$M^{-1}\dot{x} = M^{-1}Ax + M^{-1}Bu$$

$$\Rightarrow \boxed{\dot{z} = M^{-1}AMz + M^{-1}Bu} \quad \cdots ③$$

Since $y = Cx + Du$

$$\Rightarrow \boxed{y = CMz + Du} \quad \cdots ④$$

b). Take Laplace transform with zero initial conditions on ① and ②

$sX(s) = AX(s) + BU(s)$, $Y(s) = CX(s) + DU(s)$

$\Downarrow$

$(sI - A)X(s) = BU(s)$

$$TF = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D \quad \cdots ⑤,$$

From ③ and ④, is the new state equation, $\begin{cases} \dot{z} = A'z + B'u \\ y = C'z + D'u. \end{cases}$

$A' = M^{-1}AM$, $B' = M^{-1}B$. $C' = CM$, $D' = D$,

Substitute them into ⑤ for new A', B', C', D',

$$TF = C'(sI - A')^{-1}B' + D'$$

$$= CM(sI - M^{-1}AM)^{-1}M^{-1}B + D$$

$$= CM(sM^{-1}M - M^{-1}AM)^{-1}M^{-1}B + D$$

$$= CM\left[M^{-1}(sI - A)M\right]^{-1}M^{-1}B + D$$

Note
$(EFG)^{-1}$
$= G^{-1}F^{-1}E^{-1}$

$$= CM\left[M^{-1}(sI - A)^{-1}M\right]M^{-1}B + D$$

$$= C(sI - A)^{-1}B + D.$$