Final Exam

EE 602 Analytic Methods (Spring 2007)

Name: Solutions

SID: ____________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{bmatrix}$$

a). For which values of k will the system Ax=b have a unique solution? $b = [2 \quad 3 \quad 7]^T$.

b). For which values of k will the system Ax=b have an infinite number of solutions? b is same as above.

c). For **k=4,** find the LU decomposition of **A.**

Solution:

We will eliminate the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 4 & k & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{pmatrix}$$

(a) and (b) We see from this that no matter what k is there is always at least one solution (there is only a potentially 0 row in the eliminated matrix, and we get a 0 in the augmented vector). We could have seen that by inspection from the original matrix, since

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}.$$

For $k \neq 5$, the matrix has rank **3,** so there is a unique solution. For $k = 5$ the matrix has rank **2,** so there are infinitely many solutions.

(c) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ using the multipliers, and $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$ from the elimination above.

Problem 2. (25 points) Determine if $A^T A$ is positive definite for the following two matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

$A^{\mathrm{T}} A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $A^{\mathrm{T}} A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular, not p.d.

Problem 3. (25 points)

a). Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

b). Can you diagonalize it? What is the transformation matrix S so that $S^{-1}AS=D$ (where D is diagonal)?

c). Compute e^{At} for the matrix A given above.

Solution:

(a) The eigenvalues are $-1, 0, 1$ since $\boldsymbol{A}$ is triangular.

$$\lambda = -1 \text{ has } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda = 0 \text{ has } x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \lambda = 1 \text{ has } x = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}.$$

Those vectors x are the columns of S (upper triangular!).

b). $S = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ So that $S^{-1}AS = D$

and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

c). Method 1:

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} e^{-t} & -2e^{-t}+2 & 7e^{t}+3e^{-t}-10 \\ 0 & 1 & 5e^{t}-5 \\ 0 & 0 & e^{t} \end{bmatrix}$$

Method 2:

$$\begin{cases} e^{-t} = \alpha_2(-1)^2 + \alpha_1(-1) + \alpha_0 \\ 1 = \alpha_2(0)^2 + \alpha_1(0) + \alpha_0 \\ e^{t} = \alpha_2(1)^2 + \alpha_1(1) + \alpha_0 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_0 = 1 \\ \alpha_1 = \frac{1}{2}(e^{t} - e^{-t}) \\ \alpha_2 = \frac{1}{2}(e^{-t} + e^{t}) - 1 \end{cases}$$

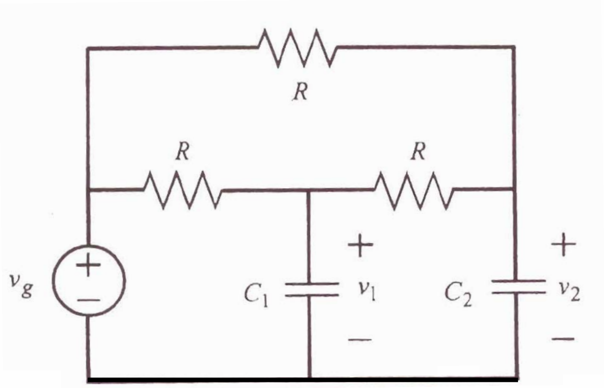
$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= I + \frac{1}{2}(e^{+t} - e^{-t}) \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}(e^{-t} + e^{t} - 2) \cdot$$

$$\begin{bmatrix} 1 & -2 & 10 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -2e^{-t} + 2 & 7e^{t} + 3e^{-t} - 10 \\ 0 & 1 & 5e^{t} - 5 \\ 0 & 0 & e^{t} \end{bmatrix}$$

Problem 4. (25 points) For the electrical circuit shown below,



its differential equation is given as:

$$\frac{v_1 - v_2}{R} + \frac{v_1 - v_g}{R} = C_1 \frac{dv_1}{dt} \qquad \frac{v_2 - v_1}{R} + \frac{v_2 - v_g}{R} = C_2 \frac{dv_2}{dt}$$

$$\dot{v}_1 = \frac{2}{C_1 R} v_1 - \frac{1}{C_1 R} v_2 - \frac{1}{C_1 R} v_g \qquad \dot{v}_2 = -\frac{1}{C_2 R} v_1 + \frac{2}{C_2 R} v_2 - \frac{1}{C_2 R} v_g$$

Consider v_g to be the input, and v_1 and v_2 to be the state variables.
a). Present the system in its state space form.
b). Find conditions on C_1 and C_2 that will make the system uncontrollable.

Solution:

$$\dot{v} = \begin{bmatrix} \frac{2}{C_1 R} & -\frac{1}{C_1 R} \\ -\frac{1}{C_2 R} & \frac{2}{C_2 R} \end{bmatrix} v + \begin{bmatrix} -\frac{1}{C_1 R} \\ -\frac{1}{C_2 R} \end{bmatrix} v_g \qquad P = \begin{bmatrix} -\frac{1}{C_1 R} & -\frac{2}{C_1^2 R^2} + \frac{1}{C_1 C_2 R} \\ -\frac{1}{C_2 R} & \frac{1}{C_1 C_2 R} - \frac{2}{C_2^2 R^2} \end{bmatrix}$$

For the system to be uncontrollable, matrix P must be rank degenerate; i.e., $\det(P) = 0$.

By some calculus, we can find that the condition $\underline{C_1 = C_2}$ leads to $\det(P) = 0$ and the system uncontrollable.