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1 A_4 has two positive eigenvalues because $a = 1$ and $ac - b^2 = 1$; $\boldsymbol{x}^{\mathrm{T}} A_1 \boldsymbol{x}$ is zero for $\boldsymbol{x} = (1, -1)$ and $\boldsymbol{x}^{\mathrm{T}} A_1 \boldsymbol{x} < 0$ for $\boldsymbol{x} = (6, -5)$.

9 $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$, rank $A = 1$, eigenvalues are 24, 0, 0, det $A = 0$.

10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots 2, $\frac{3}{2}$, $\frac{4}{3}$; $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

11 $|A_1| = 2$, $|A_2| = 6$, $|A_3| = 30$. The pivots are 2/1, 6/2, 30/6.

Finding e^A where $A = 2I_3 - J_3$ HW 8

From the $aI_n + bJ_n$ notes we get that the eigenvalues are 2, 2, −1

The matrix $M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has as its 3 columns eigenvectors

corresponding to eigenvalues 2, 2, −1, i.e.,

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Now} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\det M}\begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^t = \frac{1}{\det M}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\& \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\right) = 3$$

$$\text{So} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \text{diag}(2,2,-1)$$

So $M^{-1}AM = \text{diag}(2,2,-1)$, & this same M will diagonalize all powers of A.

So $M^{-1}e^{A}M = \text{diag}(e^2, e^2, e^{-1})$, and $e^A = M\,\text{diag}(e^2, e^2, e^{-1})M^{-1}$ or

$$e^A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{diag}(e^2, e^2, e^{-1}) \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} -e^2 & -e^2 & e^{-1} \\ e^2 & 0 & e^{-1} \\ 0 & e^2 & e^{-1} \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} (2/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} \\ -(1/3)e^2 + (1/3)e^{-1} & (2/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} \\ -(1/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} & (2/3)e^2 + (1/3)e^{-1} \end{bmatrix}$$

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Which we will check using MATLAB.

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A=[1,-1,-1;-1,1,-1;-1,-1,1]



   1  -1  -1
  -1   1  -1
  -1  -1   1

>> B=expm(A)

B =

   5.0487  -2.3404  -2.3404
  -2.3404   5.0487  -2.3404
  -2.3404  -2.3404   5.0487
```