

Problem Set 2.6

xt section will show how symmetry produces a special
lar L and U . You see P_U as the "transpose" of P_L .
LAB command `lu(pascal(4))` to produce these factors
en because the `lu` subroutine chooses the largest avail-
ill exchange rows so the second pivot is 3). But a dif-
without row exchanges. Then $[L, U] = \text{chol}(\text{pascal}(4))$
matrices as L and U . Try it.
w fifth rows do maintain $P_S = P_L P_U$:

5 35 70 for P_S 1 4 6 4 1 for P_L

row of P_L times the (same) fifth column of P_U gives
in the fifth row of P_S . The full proof of $P_S = P_L P_U$
ization can be reached in at least four different ways. I
n the course web page web.mit.edu/18.06/www, which
OpenCourseWare at ocw.mit.edu.
 P_L, P_U have so many remarkable properties—we will
te them using the Index at the end of the book.

$P_S x = b = (1, 0, 0, 0)$. This special right side means
of P_S^{-1} . That is Gauss-Jordan, matching the columns
ow the triangular P_L and P_U from 2.6 A, so we solve
bstitution) $P_U x = c$ (back substitution).

verse matrix P_S^{-1} .

ar system $P_L c = b$ is solved *top to bottom*:

$$\begin{array}{rcl} & = 1 & c_1 = +1 \\ & = 0 & c_2 = -1 \\ c_3 & = 0 & \text{gives } c_3 = +1 \\ 3c_3 + c_4 & = 0 & c_4 = -1 \end{array}$$

ation by P_L^{-1} . It produces the upper triangular system
es as always by back substitution, *bottom to top*:

$$\begin{array}{rcl} 3 + x_4 = 1 & & x_1 = +4 \\ 3 + 3x_4 = -1 & & x_2 = -6 \\ 3 + 3x_4 = 1 & \text{gives} & x_3 = +4 \\ x_4 = -1 & & x_4 = -1 \end{array}$$

P_S^{-1} has that x in its first column:

$$(4) = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

blems 1–14 compute the factorization $A = LU$ (and also $A = LDU$).

(Important) Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = b$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = c$:

$$\begin{array}{l} x + y = 5 \\ x + 2y = 7 \end{array} \rightarrow \begin{array}{l} x + y = 5 \\ y = 2 \end{array} \quad \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

That step subtracted $\ell_{21} = \underline{\hspace{1cm}}$ times row 1 from row 2. The reverse step
adds ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L = \underline{\hspace{1cm}}$.
Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get $\underline{\hspace{1cm}} = \underline{\hspace{1cm}}$.
In letters, L multiplies $Ux = c$ to give $\underline{\hspace{1cm}}$.

(Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{array}{l} x + y + z = 5 \\ x + 2y + 3z = 7 \\ x + 3y + 6z = 11 \end{array} \quad \begin{array}{l} x + y + z = 5 \\ y + 2z = 2 \\ y + 5z = 6 \end{array} \quad \begin{array}{l} x + y + z = 5 \\ y + 2z = 2 \\ z = 2 \end{array}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ in
 $Ax = b$ by subtracting $\ell_{31} = \underline{\hspace{1cm}}$ times equation 1 and $\ell_{32} = \underline{\hspace{1cm}}$ times
the *final* equation 2. Reverse that to recover $[1 \ 3 \ 6 \ 11]$ in A and b from the
final $[1 \ 1 \ 1 \ 5]$ and $[0 \ 1 \ 2 \ 2]$ and $[0 \ 0 \ 1 \ 2]$ in U and c :

Row 3 of $[A \ b] = (\ell_{31} \text{ Row 1} + \ell_{32} \text{ Row 2} + 1 \text{ Row 3})$ of $[U \ c]$.

In matrix notation this is multiplication by L . So $A = LU$ and $b = Lc$.

Write down the 2 by 2 triangular systems $Lc = b$ and $Ux = c$ from Problem 1.
Check that $c = (5, 2)$ solves the first one. Find x that solves the second one.

What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 2?
Check that $c = (5, 2, 2)$ solves the first one. Which x solves the second one?

What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to
factor A into LU :

Problem 5

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

What two elimination matrices E_{21} and E_{32} put A into upper triangular form
 $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

Problem 6

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}$$

$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

- 8 Suppose A is already lower triangular with 1's on the diagonal. Then $U = I$!

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L (nicer than E).

- 9 When zero appears in a pivot position, $A = LU$ is not possible! (We are requiring nonzero pivots in U .) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ f & h & \\ & & i \end{bmatrix}.$$

This difficulty is fixed by a row exchange. That needs a "permutation" P .

- 10 Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

- 11 What are L and D for this matrix A ? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

- 12 A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} a & b & c & d \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with U !

- 14 Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$.

Problems 15-16 use L and U (without needing A) to solve

- 15 Solve the triangular system $Lc = b$ to find c . Then so

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

For safety find $A = LU$ and solve $Ax = b$ as usual.

- 16 Solve $Lc = b$ to find c . Then solve $Ux = c$ to find x

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- 17 (a) When you apply the usual elimination steps to L ,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I , what matrix U do you get?

(c) When you apply the same steps to LU , what matrix U do you get?

- 18 If $A = LDU$ and also $A = L_1D_1U_1$ with all factors $D = D_1$ and $U = U_1$. "The factors are unique."

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the factors equal? Deduce $L = L_1$ and $U = U_1$ (they all have $D = D_1$).