Then $\lambda^2(3-\lambda)=0$ and the eigenvalues are $\lambda_1=0, \lambda_2=0, \lambda_3=3$. The eigenvectors for $\lambda=3$ are multiples of $\boldsymbol{x}_3=(1,1,1)$. The eigenvectors for $\lambda_1=\lambda_2=0$ are *any two independent vectors in the plane* $x+y+z=0$. These are the columns of all possible eigenvector matrices S:

$$S=\begin{bmatrix} x & X & c \\ y & Y & c \\ -x-y & -X-Y & c \end{bmatrix} \quad \text{and} \quad S^{-1}AS=\Lambda=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

where $c \neq 0$ and $xY \neq yX$. The powers A^n come quickly by multiplication:

$$A^2=\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}=3A \quad \text{and} \quad A^n=3^{n-1}A.$$

To find matrices B that commute with A, look at AB and BA. The 1's in A produce the column sums C_1, C_2, C_3 and the row sums R_1, R_2, R_3 of B:

$$AB=\text{column sums}=\begin{bmatrix} C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \qquad BA=\text{row sums}=\begin{bmatrix} R_1 & R_1 & R_1 \\ R_2 & R_2 & R_2 \\ R_3 & R_3 & R_3 \end{bmatrix}$$

If $AB=BA$, *all six column and row sums of B must be the same.* One possible B is A itself, since $AA=AA$. B is any linear combination of permutation matrices!

This is a 5-dimensional space (Problem 3.5.39) of matrices that commute with A. All B's share the eigenvector $(1,1,1)$. Their other eigenvectors are in the plane $x+y+z=0$. Three degrees of freedom in the λ's and two in the unit eigenvectors.

Problem Set 6.2

Questions 1–8 are about the eigenvalue and eigenvector matrices.

1 Factor these two matrices into $A=S\Lambda S^{-1}$:

$$A=\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

2 If $A=S\Lambda S^{-1}$ then $A^3=(\quad)(\quad)(\quad)$ and $A^{-1}=(\quad)(\quad)(\quad)$.

3 If A has $\lambda_1=2$ with eigenvector $\boldsymbol{x}_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2=5$ with $\boldsymbol{x}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

4 Suppose $A=S\Lambda S^{-1}$. What is the eigenvalue matrix for $A+2I$? What is the eigenvector matrix? Check that $A+2I=(\quad)(\quad)(\quad)^{-1}$.

5 True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

6 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix S is triangular, then S^{-1} is triangular. Prove that A is also triangular.

7 Describe all matrices S that diagonalize this matrix A:

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

8 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Questions 9–14 are about Fibonacci and Gibonacci numbers.

9 For the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, compute A^2 and A^3 and A^4. Then use the text and a calculator to find F_{20}.

10 Suppose each number G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

11 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

12 The numbers λ_1^k and λ_2^k satisfy the Fibonacci rule $F_{k+2} = F_{k+1} + F_k$:

$$\lambda_1^{k+2} = \lambda_1^{k+1} + \lambda_1^k \quad \text{and} \quad \lambda_2^{k+2} = \lambda_2^{k+1} + \lambda_2^k.$$

Prove this by using the original equation for the λ's. Then any combination of λ_1^k and λ_2^k satisfies the rule. The combination $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$ gives the right start $F_0 = 0$ and $F_1 = 1$.

13 Lucas started with $L_0 = 2$ and $L_1 = 1$. The rule $L_{k+2} = L_{k+1} + L_k$ is the same, so Fibonacci's matrix A is the same. Add its eigenvectors $x_1 + x_2$:

$$\begin{bmatrix}\lambda_1\\1\end{bmatrix} + \begin{bmatrix}\lambda_2\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(1+\sqrt{5})\\1\end{bmatrix} + \begin{bmatrix}\frac{1}{2}(1-\sqrt{5})\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}L_1\\L_0\end{bmatrix}.$$

After 10 steps the second component of $A^{10}(x_1+x_2)$ is $\lambda_1^{10}+\lambda_2^{10}$. Compute that Lucas number L_{10} by $L_{k+2} = L_{k+1} + L_k$, and compute approximately by λ_1^{10}.

14 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 15–18 are about diagonalizability.

15 True or false: If the eigenvalues of A are 2, 2, 5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

16 True or false: If the only eigenvectors of A are multiples of (1, 4) then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $S\Lambda S^{-1}$.

17 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $Ax = 5x$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix}8 & \\ & 2\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}9 & 4\\ & 1\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}10 & 5\\-5 & \end{bmatrix}$$

18 The matrix $A = \left[\begin{smallmatrix}3 & 1\\0 & 3\end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is ____. Change one entry to make A diagonalizable. Which entries could you change?

Questions 19–23 are about powers of matrices.

19 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Which of these matrices has $A^k \to 0$?

$$A = \begin{bmatrix}.6 & .4\\.4 & .6\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}.6 & .9\\.1 & .6\end{bmatrix}.$$

20 (Recommended) Find Λ and S to diagonalize A in Problem 19. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ____.

21 Find Λ and S to diagonalize B in Problem 19. What is $B^{10}u_0$ for these u_0?

$$u_0 = \begin{bmatrix}3\\1\end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix}3\\-1\end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix}6\\0\end{bmatrix}.$$