Which one starts with $y(0) = 1$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -y$ produces a vector equation $u' = Au$:

$$u = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au.$$

Put $y(t)$ from part (a) into $u(t) = (y, y')$. This solves Problem 11 again.

13 A particular solution to $du/dt = Au - b$ is $u_p = A^{-1}b$, if A is invertible. The solutions to $du/dt = Au$ give u_n. Find the complete solution $u_p + u_n$ to

(a) $\dfrac{du}{dt} = 2u - 8$ (b) $\dfrac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$.

14 If c is not an eigenvalue of A, substitute $u = e^{ct}v$ and find v to solve $du/dt = Au - e^{ct}b$. This $u = e^{ct}v$ is a particular solution. How does it break down when c is an eigenvalue?

15 Find a matrix A to illustrate each of the unstable regions in Figure 6.4:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$

(b) $\lambda_1 > 0$ and $\lambda_2 > 0$

(c) Complex λ's with real part $a > 0$.

Questions 16–25 are about the matrix exponential e^{At}.

16 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}u_0$ solves $u' = Au$.

17 The matrix $B = \left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right]$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

18 Starting from $u(0)$ the solution at time T is $e^{AT}u(0)$. Go an additional time t to reach $e^{At}(e^{AT}u(0))$. This solution at time $t + T$ can also be written as ____. Conclusion: e^{At} times e^{AT} equals ____.

19 Write $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

20 If $A^2 = A$ show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ in Problem 19 this gives $e^{At} =$ ____.

21 Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 19–20 and 17:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

22 Write $A = \begin{bmatrix}1&1\\0&3\end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check e^{At} when $t=0$.

23 Put $A = \begin{bmatrix}1&3\\0&0\end{bmatrix}$ into the infinite series to find e^{At}. First compute A^2!

$$e^{At} = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}t&3t\\0&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\ &\ \\ \ &\ \end{bmatrix} + \cdots = \begin{bmatrix}e^t&\ \\0&\ \end{bmatrix}.$$

24 Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write down its inverse.

(b) Write down its eigenvalues. If $Ax = \lambda x$ then $e^{At}x = \_\_\_ x$.

25 Find a solution $x(t), y(t)$ that gets large as $t \to \infty$. To avoid this instability a scientist exchanged the two equations:

$$\begin{aligned} dx/dt &= 0x - 4y \\ dy/dt &= -2x + 2y \end{aligned} \quad \text{becomes} \quad \begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

Now the matrix $\begin{bmatrix}-2&2\\0&-4\end{bmatrix}$ is stable. It has negative eigenvalues. Comment on this.

SYMMETRIC MATRICES ■ 6.4

For projection onto a line, the eigenvalues are 1 and 0. Eigenvectors are *on* the line (where $Px = x$) and *perpendicular* to the line (where $Px = \mathbf{0}$). Now we open up to all other ***symmetric matrices***. It is no exaggeration to say that these are the most important matrices the world will ever see—in the theory of linear algebra and also in the applications. We come immediately to the key questions about symmetry. Not only the questions, but also the answers.

What is special about $Ax = \lambda x$ when A is symmetric? We are looking for special properties of the eigenvalues λ and the eigenvectors x when $A = A^{\mathrm{T}}$.

The diagonalization $A = S\Lambda S^{-1}$ will reflect the symmetry of A. We get some hint by transposing to $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\Lambda S^{\mathrm{T}}$. Those are the same since $A = A^{\mathrm{T}}$. Possibly S^{-1} in the first form equals S^{T} in the second form. Then $S^{\mathrm{T}}S = I$. That makes each eigenvector in S orthogonal to the other eigenvectors. The key facts get first place in the Table at the end of this chapter, and here they are:

1. A symmetric matrix has only ***real eigenvalues***.
2. The ***eigenvectors*** can be chosen ***orthonormal***.

Its eigenvector matrix S becomes an orthogonal matrix Q. Ort
$Q^{-1} = Q^{\mathrm{T}}$—what we suspected about S is true. To remembe
when we choose orthonormal eigenvectors.

Why do we use the word "choose"? Because the eigenvec
unit vectors. Their lengths are at our disposal. We will choose un
of length one, which are orthonormal and not just orthogonal. T
its special and particular form $Q\Lambda Q^{\mathrm{T}}$ for symmetric matrices:

6H (Spectral Theorem) Every symmetric matrix has the fact
with real eigenvalues in Λ and orthonormal eigenvectors in Q:

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}$$

It is easy to see that $Q\Lambda Q^{\mathrm{T}}$ is symmetric. Take its transpose.
which is $Q\Lambda Q^{\mathrm{T}}$ again. The harder part is to prove that every
real λ's and orthonormal x's. This is the *"spectral theorem"* i
"principal axis theorem" in geometry and physics. We have to
will approach the proof in three steps:

1. By an example (which only proves that $A = Q\Lambda Q^{\mathrm{T}}$ migh

2. By calculating the 2 by 2 case (which convinces most fai

3. By a proof when no eigenvalues are repeated (leaving onl

The diehards are worried about repeated eigenvalues. Are the
eigenvectors? *Yes, there are*. They go into the columns of S (wh
last page before the problems outlines this fourth and final step

We now take steps 1 and 2. In a sense they are optiona
mostly for fun, since it is included in the final n by n case.

Example 1 Find the λ's and x's when $A = \begin{bmatrix}1&2\\2&4\end{bmatrix}$ and $A - \lambda$

Solution The equation $\det(A - \lambda I) = 0$ is $\lambda^2 - 5\lambda = 0$. The ei
(*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue
and $\lambda = 5$ is the other eigenvalue so that $0 + 5$ agrees with 1
down the diagonal of A.

Two eigenvectors are $(2, -1)$ and $(1, 2)$—orthogonal but no
eigenvector for $\lambda = 0$ is in the *nullspace* of A. The eigenvector
umn space. We ask ourselves, why are the nullspace and colum
The Fundamental Theorem says that the nullspace is perpendicu