Midterm Exam

EE 602 Analytic Methods (Fall 2009)

Name: Solutions

SID: ____________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Find the determinant of the following matrix by reducing it to a triangular one.

$$\begin{bmatrix} -3 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ -6 & -2 & -4 & 2 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -3 & 0 & 1 & 1 \\ 3 & 1 & 2 & 2 \\ -6 & -2 & -4 & 2 \\ 1 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & -2 & -6 & 0 \\ 0 & -1 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & -1 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & \frac{10}{3} & \frac{7}{3} \end{bmatrix}$$

$$\xrightarrow[A_1]{} \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & \frac{10}{3} & 6+\frac{7}{3} \\ 0 & 0 & \frac{10}{3} & \frac{7}{3} \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & \frac{10}{3} & 6+\frac{7}{3} \\ 0 & 0 & 0 & -6 \end{bmatrix} \triangleq A_1$$

$$\det A = \det A_1 = (-3)\cdot(1)\cdot\left(\frac{10}{3}\right)\cdot(-6) = 60.$$

Problem 2. (25 points) Find LU decomposition of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -6 & -6 \\ -7 & -7 & 9 \end{bmatrix}$$

Solution

Solution. We reduce A to row echelon form with elementary matrices. First, we replace row two by row two minus 4 times row one. The elementary matrix required is

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{thus} \quad E_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -14 & -18 \\ -7 & -7 & 9 \end{bmatrix}.$$

Then we replace row three by row three plus 7 times row one. Thus

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2(E_1 A) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -14 & -18 \\ 0 & 7 & 30 \end{bmatrix}.$$

Next, we replace row three by row three minus $-\frac{1}{2}$ times row two. (We say "minus," not "plus.") We need

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \quad \text{thus} \quad E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -14 & -18 \\ 0 & 0 & 21 \end{bmatrix} = U.$$

Since $(E_3 E_2 E_1)A = U$, $A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U$, so

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -7 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Thus $A = LU$;

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -6 & -6 \\ -7 & -7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -7 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -14 & -18 \\ 0 & 0 & 21 \end{bmatrix}.$$

an equation that should be checked (of course).

Problem 3. (25 points) Write the complete solutions $x=x_p+x_n$ to the system as in the following equation:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.$$

Note that x_p is the particular solution, and x_n is the homogenous solution.

Solution:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow{E_1\bar{A}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{E_2\bar{A}_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_3\bar{A}_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \triangleq \bar{A}_3$$

$$\bar{A}_3F = \bar{A}_3\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_3E_2E_1AF\cdot F^Tx = E_3E_2E_1b$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} w \\ y \\ \hline x \\ z \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \\ \hline 0 \end{array}\right]$$

Two free-choosing variables x & z: $\begin{bmatrix} w \\ y \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} w \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}x + \begin{bmatrix} 1 \\ -1 \end{bmatrix}z = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So $$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}z$$

Problem 4. (25 points) Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

(Hint: The row space consists of all linear combinations of the rows.)

Solution: $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & -3 & -6 & -3 & -21 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & -\frac{2}{3} & 19-\frac{5\times 13}{3} \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & -\frac{2}{3} & -\frac{8}{3} \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space of A is

$[1 \ 1 \ 3 \ 1 \ 6]$, $[0 \ -3 \ -6 \ -1 \ -13]$,

$[0 \ 0 \ 0 \ 2 \ 8]$.

or any three linear independent row vectors of A.