

# **EE602 Analytic Methods**

## **Lecture 1: Matrix Algebra**

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# Matrices

- **A Matrix:**

Rectangular arrays  $A = [ a_{i,j} ]$  of elements  $a_{i,j}$  (the first index refers to the row and the 2<sup>nd</sup> to the column)

- The **order** of  $A$  is  $n \times m$  if it has  $n$  rows and  $m$  columns.

Remark: Usually we simplify the notation and write  $a_{ij}$  instead of  $a_{i,j}$  when we are displaying literals. However reference to a specific element may need a comma as  $a_{123}$  could mean  $a_{12,3}$  or  $a_{1,23}$ .

# Matrices

- The **transpose** is obtained by interchanging the rows and columns, eg:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 7 \end{bmatrix}$$

# Special Matrices

- Square matrix (of order  $n$ ): the number of rows is equal to the number of the columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

# Identity Matrix

- Identity matrix: a square matrix with the diagonal elements 1 and all other elements 0

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

# Zero Matrix

- Zero matrix: all elements are 0

$$0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

# Hermitian Matrix

- Hermitian (or self-adjoint) matrix: a square matrix that the elements on the main diagonal are real and whenever two elements are positioned symmetrically with respect to the main diagonal, they're mutually complex conjugate

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \bar{a}_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

# Lower-Triangular Matrix

- Lower-triangular matrix: a square matrix whose elements above the main diagonal are zeros

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

# Upper-Triangular Matrix

- Upper-triangular matrix: a square matrix whose elements below the main diagonal are zeros

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

# Diagonal Matrix

- Diagonal matrix: a square matrix that all elements lying outside of the main diagonal are zeros

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

# Symmetric Matrix

- Symmetric matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

# Vectors

- Vector:
  - Column vector:  $n \times 1$  matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- Row vector:  $1 \times n$  matrix

$$b^T = [b_1 \quad b_2 \quad \cdots \quad b_n]$$

# Matrix Operations

- Addition: The sum of two matrices  $A+B$  is defined as  
$$A+B = [ a_{i,j} + b_{i,j} ]$$
- Properties:
  - **Associative:**  $A+(B+C) = (A+B)+C$
  - **Commutative:**  $A+B = B+A$

# Matrix Operations

- Multiplication by a scalar  $\alpha A$ , where  $\alpha$  is a scalar, is defined as  $[\alpha a_{i,j}]$
- Properties:
  - **Distributive** 2 ways:  
 $(\alpha+\beta)A = \alpha A + \beta A$   
 $\alpha(A+B) = \alpha A + \alpha B.$

# Matrix Operations

- Vector multiplication:

if  $\mathbf{x} = [x_1, x_2, x_3]^T$  and  $\mathbf{y} = [y_1, y_2, y_3]^T$  are 2 column vectors, then:

$$\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

# Matrix Operations

- Matrix multiplication:

$$AB = \begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} b_{ij} \end{bmatrix} = \begin{bmatrix} c_{ij} \end{bmatrix} \text{ where } c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

*eg.*  $\begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 23 & 14 & 27 \\ 15 & 18 & 14 & 26 \end{bmatrix}$

# Matrix Operations

- Properties of matrix multiplication:
  - **Associative:**  $A(BC) = (AB)C$
  - **Distributive:**  $A(B+C) = AB + AC$
  - With an **identity matrix:**  $AI = A = IA$
  - **NOT commutative:**  $AB \neq BA$

e.g. Does  $\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} ???$

Hint : What are the sizes of the 2 matrix products ?

e.g.  $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$  BUT  $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

# Special Properties

- If  $A = A^T$  , then we say  $A$  is **symmetric**
- If  $A = -A^T$  , then we say  $A$  is **skew-symmetric**  
e.g.  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- A square matrix is called **nilpotent** if there's a positive integer  $k$  such that  $A^k = 0$   
 $k$  is the degree of nilpotency of  $A$ .

# Determinants

- Definition:

A **minor** of order  $n-1$  is defined to be the determinant of a sub-matrix of  $A$  obtained by deleting one row and one column from  $A$ , denoted as  $M_{ij}$  (i.e. deleting the  $i$ th row and  $j$ th column)

- **Cofactor expansions:** Let  $A$  be a square matrix of order  $n$ , then for any  $i, j$  ( $1 \leq i, 1 \leq j$ ),

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

or,  $\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$

where  $A_{pq} = (-1)^{p+q} M_{pq}$  (called cofactors).

# Determinants

- A example of cofactor expansions:

$$A = \begin{bmatrix} 3 & 12 & -1 & 0 \\ -1 & 0 & 5 & 6 \\ 0 & 3 & 0 & 0 \\ 0 & 4 & -1 & 2 \end{bmatrix}$$

The determinant of A can be easily obtained by use of cofactor expansion along the third row:

$$\det A = 3A_{32} = 3(-1)^{3+2} \det \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & 6 \\ 0 & -1 & 2 \end{bmatrix} = -138$$

# Laplace's Theorem

- Definition:

For a  $m \times n$  matrix  $A$ , **minor of order  $p$**  of  $A$  is the determinant of a sub-matrix obtained by deleting  $m-p$  rows and  $n-p$  columns of  $A$ .

$A \begin{pmatrix} i_1, i_2 \cdots i_p \\ j_1, j_2 \cdots j_p \end{pmatrix}$  means the determinant of the submatrix obtained by retaining rows  $i_1, i_2 \cdots i_p$  and columns  $j_1, j_2 \cdots j_p$

# Laplace's Theorem

- Definition:

**Complementary minor:** the determinant of a submatrix obtained by deleting the rows and the columns of  $A$  listed in the bracket.

$A \begin{pmatrix} i_1, i_2 \cdots i_p \\ j_1, j_2 \cdots j_p \end{pmatrix}^c$  means the submatrix obtained by deleting rows  $i_1, i_2 \cdots i_p$  and columns  $j_1, j_2 \cdots j_p$

# Laplace's Theorem

- Definition:

**Complementary cofactor:**

$$A^c \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} = (-1)^s A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}^c$$

where  $s = (i_1 + i_2 + \cdots + i_p) + (j_1 + j_2 + \cdots + j_p)$ .

An example:

Let  $A = [a_{ij}]_{i,j=1}^5$ , then

$$A \begin{pmatrix} 2,3,5 \\ 1,2,4 \end{pmatrix} = \det \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{51} & a_{52} & a_{54} \end{bmatrix} = A \begin{pmatrix} 2,4 \\ 3,5 \end{pmatrix}^C$$

$$A \begin{pmatrix} 2,3,5 \\ 1,2,4 \end{pmatrix}^C = \det \begin{bmatrix} a_{13} & a_{15} \\ a_{43} & a_{45} \end{bmatrix} = A \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}$$

$$A^C \begin{pmatrix} 2,3,5 \\ 1,2,4 \end{pmatrix} = (-1)^s \det \begin{bmatrix} a_{13} & a_{15} \\ a_{43} & a_{45} \end{bmatrix} = -(a_{13}a_{45} - a_{15}a_{43}),$$

where  $s = (2 + 3 + 5) + (1 + 2 + 4) = 17$ .

# Laplace's Theorem

- Laplace's Theorem:

Let  $A$  denote an arbitrary  $n \times n$  matrix and let any  $p$  rows (or columns) of  $A$  be chosen. Then the determinant of  $A$  is equal to the sum of the products of all  $C_{n,p}$  minors lying in these rows with their corresponding complementary cofactors:

$$\det A = \sum_j A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} A^c \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}$$

where the summation over all  $C_{n,p}$  distinct sets of (column) indices  $j_1, j_2, \dots, j_p$  ( $1 \leq j_1 \leq j_2 \leq \dots \leq j_p \leq n$ ). Or, equivalently, using columns,

$$\det A = \sum_i A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} A^c \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}$$

where  $1 \leq i_1 \leq i_2 \leq \dots \leq i_p \leq n$ .

# Laplace's Theorem

- Example: Do  $\det \begin{bmatrix} \begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \end{bmatrix}$  by Laplace using rows 1 & 2

Solution: For each choice (s,t) of the possible sets of 2 columns, we must evaluate the term

$$t(s, t) = \det \begin{bmatrix} A \begin{pmatrix} 1,2 \\ s,t \end{pmatrix} \end{bmatrix} \det \begin{bmatrix} A \begin{pmatrix} 1,2 \\ s,t \end{pmatrix}^c \end{bmatrix} (-1)^{1+2+s+t} \text{ Put } A \begin{pmatrix} 1,2 \\ s,t \end{pmatrix} \text{ green \& } A \begin{pmatrix} 1,2 \\ s,t \end{pmatrix}^c \text{ red}$$

For example when (s,t)=(1,2)

$$\text{green} \rightarrow \begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \leftarrow \text{red}$$

	(s,t)	Sign	G × R	t(s,t)
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(1,2)	$(-1)^{1+2+1+2}$ =1	$(-2+3) \times$ $(-2+3) =$ 1	+1
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(1,3)	$(-1)^{1+2+1+3}$ =-1	$(2-2) \times$ $(-2-1)$ =0	0
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(1,4)	$(-1)^{1+2+1+4}$ =1	$(4-5) \times$ $(-6-2) =$ 8	+8
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(2,3)	$(-1)^{1+2+2+3}$ =1	$(-3+2) \times$ $(-3-1) =$ 4	+4
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(2,4)	$(-1)^{1+2+2+4}$ =-1	$(-6+5) \times$ $(-9-2) =$ 11	-11
$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$	(3,4)	$(-1)^{1+2+3+4}$ =1	$(4-5) \times$ $(3-2) =$ -1	-1



is red



green

TOTAL +1

# Binet-Cauchy Formula

- For an  $n \times m$  matrix  $A$  and an  $m \times n$  matrix  $B$  ( $n \leq m$ )

$$\det(AB) = \sum_{\substack{\text{all } j_1 < j_2 < \dots < j_n \\ \text{in } (1, 2, \dots, m)}} \det \begin{bmatrix} A \begin{pmatrix} 1, 2, \dots, n \\ j_1, j_2, \dots, j_n \end{pmatrix} \end{bmatrix} \det \begin{bmatrix} B \begin{pmatrix} j_1, j_2, \dots, j_n \\ 1, 2, \dots, n \end{pmatrix} \end{bmatrix}$$

all  $j_1 < j_2 < \dots < j_n$  in  $(1, 2, \dots, m)$  is another way of saying

all possible  $\binom{m}{n}$  choices for the  $j_1, j_2, \dots, j_n$

- Example:

$$A = \begin{bmatrix} 2 & 1 & -3 & 2 \\ -1 & 2 & -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -3 \\ 2 & -1 \\ 1 & -3 \\ 2 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Solution: Find all  $2 \times 2$ 's in A and multiply each by its corresponding  $2 \times 2$  in B.

$$\begin{aligned} \det(AB) &= \det \begin{bmatrix} A(1,2) \\ A(1,2) \end{bmatrix} \det \begin{bmatrix} B(1,2) \\ B(1,2) \end{bmatrix} + \det \begin{bmatrix} A(1,2) \\ A(1,3) \end{bmatrix} \det \begin{bmatrix} B(1,3) \\ B(1,2) \end{bmatrix} + \det \begin{bmatrix} A(1,2) \\ A(1,4) \end{bmatrix} \det \begin{bmatrix} B(1,4) \\ B(1,2) \end{bmatrix} \\ &+ \det \begin{bmatrix} A(1,2) \\ A(2,3) \end{bmatrix} \det \begin{bmatrix} B(2,3) \\ B(1,2) \end{bmatrix} + \det \begin{bmatrix} A(1,2) \\ A(2,4) \end{bmatrix} \det \begin{bmatrix} B(2,4) \\ B(1,2) \end{bmatrix} + \det \begin{bmatrix} A(1,2) \\ A(3,4) \end{bmatrix} \det \begin{bmatrix} B(3,4) \\ B(1,2) \end{bmatrix} \\ &= \quad (5)(6) \quad \quad \quad + (-9)(3) \quad \quad \quad + (10)(6) \\ &\quad + (3)(-5) \quad \quad \quad + (0)(2) \quad \quad \quad + (-6)(6) = \end{aligned}$$

# Binet-Cauchy Formula

- Special case:

A & B both square, then

$$\det(AB) = \det(A) \det(B).$$