Final Exam
EE 602 Analytic Methods (Spring 2009)

Name: ________________
SID: ________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Given two functions \( f = x^T A x \),

\[
\begin{align*}
f_1 &= 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3), \\
f_2 &= 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3),
\end{align*}
\]

a) Find the symmetric matrix \( A \) for each of the above two functions.
b) For each of the matrix \( A \) found above, is it positive definite?

Solution:

\[
A_1 = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\]

Points: 2, \( \frac{3}{2}, \frac{4}{3} \)

p.d.

b) \( A_2 = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix} \) not p.d.
Problem 2. (25 points) Suppose using the state variable $x$ we have

\[
\dot{x} = Ax + Bu
\]
\[
y = Cx + Du
\]

Consider a change of variables $x = Mz$ where $M$ is non-singular.

a) What are the new state equations?

b) Does the controllability of the system change? Prove your answer.

Solution: (a).

\[
x = Mz, \quad \dot{z} = M^{-1}x
\]
\[
\dot{z} = M^{-1}AMz + M^{-1}Bu
\]
\[
= A'z + B'u
\]
\[
y = CMz + Du \triangleq C'z + D'u
\]

b). Controllability matrix for original system:

\[
P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}
\]

Controllability matrix for new system:

\[
P' = \begin{bmatrix} B' & A'B' & \cdots & A'^{n-1}B' \end{bmatrix}
\]
\[
= [M^{-1}B \quad M^{-1}AMM^{-1}B \quad \cdots \quad (M^{-1}AM)^{n-1}M^{-1}B]
\]
\[
= [M^{-1}B \quad M^{-1}AB \quad \cdots \quad M^{-1}A^{n-1}B]
\]

Since $M$ is nonsingular,

\[
\text{rank}(P') = \text{rank}(P)
\]

so the controllability does not change.
Problem 3. (25 points) Multinational companies in the Americas, Asia, and Europe have assets of $4 trillion. At the start, $2 trillion are in the Americas and $2 trillion in Europe. Each year $\frac{1}{2}$ the American money stays home, and $\frac{1}{4}$ goes to each of Asia and Europe. For Asia, and Europe, $\frac{1}{2}$ stays home and $\frac{1}{2}$ is sent to Americas.

a) Find the matrix that gives

$$
\begin{bmatrix}
\text{Americas} \\
\text{Asia} \\
\text{Europe}
\end{bmatrix}_{\text{year } k+1} = A
\begin{bmatrix}
\text{Americas} \\
\text{Asia} \\
\text{Europe}
\end{bmatrix}_{\text{year } k}
$$

b) Find the eigenvalues and eigenvectors of A.

c) Find the distribution of the $4$ trillion at year $k$.

d) Find the distribution of the $4$ trillion as the world ends.

Solutions:

a) \( A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & 0 \\
\frac{1}{4} & 0 & \frac{1}{2}
\end{pmatrix} \)

b) \( \lambda = 1, \ 1, \ 0 \)

\( x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)
\( x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \)
\( x_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \)

c) \( \begin{pmatrix} 2 \\ 1 - (\frac{1}{2})^k \\ 1 + (\frac{1}{2})^k \end{pmatrix} \)

d) \( \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \)
Problem 4. (25 points) For the skew-symmetric equation
\[
\frac{du}{dt} = Au = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
\]

a) Write out \( \dot{u}_1, \dot{u}_2, \dot{u}_3 \), and confirm that \( \dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 = 0 \).

b) Deduce that \( u_1^2 + u_2^2 + u_3^2 = 0 \).

c) Find the eigenvalues of A.

Solutions: a)
\[
\dot{u}_1 = cu_2 - bu_3 \\
\dot{u}_2 = -cu_1 + au_3 \\
\dot{u}_3 = bu_1 - au_2
\]

so \( \dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 = 0 \)

b)
\[
\frac{d}{dt} (u_1^2 + u_2^2 + u_3^2) = 2u_1 \dot{u}_1 + 2u_2 \dot{u}_2 + 2u_3 \dot{u}_3 = 0
\]

\( \Rightarrow \) \( u_1^2 + u_2^2 + u_3^2 \) is a constant.

It is 0 when \( u_1(0) = u_2(0) = u_3(0) = 0 \).

c) \( \lambda = 0 \), and \( \lambda = \pm j \sqrt{a^2 + b^2 + c^2} \).