Final Exam

EE 602 Analytic Methods (Spring 2009)

Name: Solutions

SID: ____________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Given two functions $f = x^T A x$,

$$f_1 = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3),$$

$$f_2 = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3),$$

a) Find the symmetric matrix A for each of the above two functions.
b) For each of the matrix A found above, is it positive definite?

Solution:

a). $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ pivots $2, \frac{3}{2}, \frac{4}{3}$.

p.d.

b). $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ not p.d.

Problem 2. (25 points) Suppose using the state variable **x** we have

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Consider a change of variables **x** = M**z** where M is non-singular.

a) What are the new state equations?

b) Does the controllability of the system change? Prove your answer.

Solution: a).

$$x = Mz, \quad z = M^{-1}x$$

$$\dot{z} = M^{-1}AMz + M^{-1}Bu$$

$$\triangleq A'z + B'u$$

$$y = CMz + Du \triangleq C'z + D'u$$

b). controllability matrix for original system:

$$P = [B \ AB \ \cdots \ A^{n-1}B]$$

Controllability matrix for new system:

$$P' = [B' \ A'B' \ \cdots \ A'^{n-1}B']$$

$$= [M^{-1}B \ M^{-1}AMM^{-1}B \ \cdots \ (M^{-1}AM)^{n-1}M^{-1}B]$$

$$= [M^{-1}B \ M^{-1}AB \ \cdots \ M^{-1}A^{n-1}B]$$

$$= M^{-1}P$$

Since M is nonsingular,

$$\text{rank}(P') = \text{rank}(P)$$

So the controllability does not change.

Problem 3. (25 points) Multinational companies in the Americas, Asia, and Europe have assets of \$4 trillion. At the start, \$2 trillion are in the Americas and \$2 trillion in Europe. Each year ½ the American money stays home, and ¼ goes to each of Asia and Europe. For Asia, and Europe, ½ stays home and ½ is sent to Americas.

a) Find the matrix that gives

$$\begin{bmatrix} Americas \\ Asia \\ Europe \end{bmatrix}_{year\ k+1} = A \begin{bmatrix} Americas \\ Asia \\ Europe \end{bmatrix}_{year\ k}$$

b) Find the eigenvalues and eigenvectors of A.

c) Find the distribution of the \$4 trillion at year *k*.

d) Find the distribution of the \$4 trillion as the world ends.

Solutions: a). $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$

b). $\lambda = 1, \frac{1}{2}, 0.$

$X_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

c). $\begin{bmatrix} 2 \\ 1-(\frac{1}{2})^k \\ 1+(\frac{1}{2})^k \end{bmatrix}.$

d). $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$

Problem 4. (25 points) For the skew-symmetric equation

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

a) Write out $\dot{u}_1, \dot{u}_2, \dot{u}_3$, and confirm that $\dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 = 0$.

b) Deduce that $u_1^2 + u_2^2 + u_3^2 = 0$.

c) Find the eigenvalues of A.

Solutions: a). $\dot{u}_1 = c u_2 - b u_3$,

$\dot{u}_2 = -c u_1 + a u_3$

$\dot{u}_3 = b u_1 - a u_2$

So $\dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 = 0$

b). $\frac{d(u_1^2 + u_2^2 + u_3^2)}{dt} = 2u_1 \cdot \dot{u}_1 + 2u_2 \dot{u}_2 + 2u_3 \dot{u}_3 = 0$

$\Rightarrow u_1^2 + u_2^2 + u_3^2$ is a constant.

it is 0 when $u_1(0) = u_2(0) = u_3(0) = 0$.

c). $\lambda = 0$, and $\lambda = \pm j\sqrt{a^2 + b^2 + c^2}$.