Final Exam

EE 602 Analytic Methods (Fall 2007)

Name: Solution

SID: ______________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (20 points) Diagonalize B and prove this formula for B^k:

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$B^k = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}$$

$$B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}.$$

Problem 2. (20 points) For $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_2x_3)$, find a 3 x 3 symmetric matrix A such that $f = x^T Ax$, and check whether A is positive definite.

$\boxed{\text{Solution}}$ The matrix A is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

To show that this is positive definite, we only need to find all the pivots:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix}.$$

The pivots are all positive, so A is positive definite.

Problem 3. (20 points) A door is opened between rooms that hold v(0)=30 people and w(0) =10 people. The movement between rooms is proportional to the difference v-w:

$$\frac{dv}{dt} = w - v, \quad \frac{dw}{dt} = v - w.$$

1) Show that the total v+w is constant (40 people).
2) Represent the system using state space equations dx/dt=Ax taking the state $x=[v \;\; w]^T$ (Note that there is no control in this case).
3) Find the eigenvalues and eigenvectors of matrix A.
4) What are v and w at t=1?

$d(v+w)/dt = dv/dt + dw/dt = (w-v) + (v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = -2$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} \\ w(1) = 20 - 10e^{-2} \end{matrix}$.

Problem 4. (20 points) True or false, provide your reasons:

1) If A is similar to B then A^2 is similar to B^2.
2) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.
3) The vectors (1,-3,2) and (2,1,-3) and (-3,2,1) are linearly independent.
4) The vectors (1,2,2), (-1,2,1), (0,8,6) are a basis for R^3.

① True $\quad A=T^{-1}BT \Rightarrow A^2=A\cdot A=(T^{-1}BT)(T^{-1}BT)=T^{-1}B^2T$

②. True $\quad$ find a T.

③ False

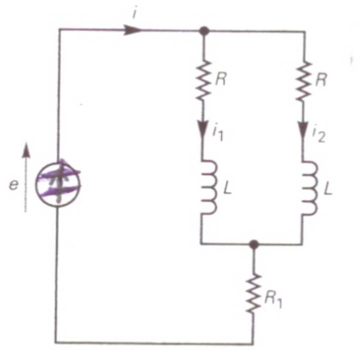
$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}; A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ columns add to } \mathbf{0}.$$

④. False

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 8 \\ 0 & 3 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 5. (20 points) Represent the following circuit in state space taking $[i_1\ i_2]^T$ as the state and e (the ~~current~~ voltage source) as the input. Is it controllable?



$$Ri_1 + L\frac{di_1}{dt} + R_1(i_1 + i_2) = e$$

$$Ri_2 + L\frac{di_2}{dt} + R_1(i_1 + i_2) = e$$

These equations can be written in the standard state format as

$$\frac{di_1}{dt} = -\left[\frac{R + R_1}{L}\right] i_1 - \frac{R_1}{L} i_2 + \frac{1}{L} e$$

$$\frac{di_2}{dt} = -\frac{R_1}{L} i_1 - \left[\frac{R + R_1}{L}\right] i_2 + \frac{1}{L} e$$

The matrices **A** and **B** of these state equations are given by

$$\mathbf{A} = \begin{bmatrix} -\frac{R + R_1}{L} & -\frac{R_1}{L} \\ -\frac{R_1}{L} & -\frac{R + R_1}{L} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{L} \\ \frac{1}{L} \end{bmatrix}$$

Hence

$$\mathbf{AB} = \begin{bmatrix} -\frac{R + 2R_1}{L^2} \\ -\frac{R + 2R_1}{L^2} \end{bmatrix}$$

and the controllability matrix of (10-83) is given by

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} +\frac{1}{L} & -\frac{R + 2R_1}{L^2} \\ +\frac{1}{L} & -\frac{R + 2R_1}{L^2} \end{bmatrix}$$

not controllable.