Final Exam

EE 602 Analytic Methods (Spring 2012)

Name: Solutions

SID: ______________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Find the limit values of y_k and z_k $(k \to \infty)$ if

$$y_{k+1} = 0.8y_k + 0.3z_k, \quad y_0 = 0,$$
$$z_{k+1} = 0.2y_k + 0.7z_k, \quad z_0 = 5.$$

Solution:

$$\begin{bmatrix} y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} y_k \\ z_k \end{bmatrix} \triangleq A \begin{bmatrix} y_k \\ z_k \end{bmatrix}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

To find A^k: eigenvalues of A: $\det(\lambda I - A) \Rightarrow \lambda = 1, 0.5$.

Eigenvectors of A: $\lambda_1 X_1 = AX_1 \Rightarrow (\lambda_1 I - A)X_1 = 0 \Rightarrow X_1 = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$

$\lambda_2 X_2 = AX_2 \Rightarrow (\lambda_2 I - A)X_2 = 0 \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Since $A = T\Lambda T^{-1} \Rightarrow A^k = \begin{bmatrix} 1 & 1 \\ \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{2}{3} & -1 \end{bmatrix}^{-1}$

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 1 & 1 \\ \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{2}{3} & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

$$\lim_{k\to\infty} \begin{bmatrix} y_k \\ z_k \end{bmatrix} = \lim_{k\to\infty} A^k \cdot \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Problem 2. (25 points) True or False. Please provide a brief reason, or provide a counterexample for the "False".

1) If an n by n matrix is diagonalizable, then it must have n different eigenvalues.
2) If A is a square matrix and det(A)=0, then A must have a row of 0s.
3) If A is positive definite, then A^{-1} is positive definite.
4) A positive definite matrix may have a negative number on its diagonal.
5) A symmetric matrix with a positive determinant is positive definite.

Solution: 1). False. Distinct eigenvalue is the sufficient but not necessary condition for diagonalization.

2). False. Conterexample: $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

det (A)=0, but A does not have a row of 0s.

3). True. $X^T A^{-1} X = (A^{-1}X)^T A (A^{-1}X) > 0$ for all $X \neq 0$

(as A is symetric, A^{-1} is symetric. A is positive definite)

4). False. (any counterexample is fine.)

5). False. Counterexample: $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ $\det(A) = 1 > 0$

but A is not positive definite

It needs all upper determinants to be positive.

Problem 3. (25 points) Find the solutions of the following differential equations with constant coefficients:

$$\begin{aligned}\dot{x}_1 &= 2x_1\\ \dot{x}_2 &= 11x_1 + 4x_2 + 5x_3\\ \dot{x}_3 &= 7x_1 - x_2 - 2x_3\end{aligned}$$

if the initial condition is $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, where c_1, c_2, c_3 are constants.

Solution: 1).

$$\dot{X} = \begin{bmatrix} 2 & 0 & 0 \\ 11 & 4 & 5 \\ 7 & -1 & -2 \end{bmatrix} X$$

where $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$.

2). $X(t) = e^{At} X(0)$

$\det(A) = 0 \Rightarrow$ eigenvalues of A are $2, 3, -1$.

$A \cancel{X}\xi_i = \lambda_i \cancel{X}\xi_i$, $i = 1, 2, 3 \Rightarrow$ eigenvectors of A are

$$\cancel{X}\xi_1 = \begin{bmatrix} 3 \\ -79 \\ 25 \end{bmatrix}, \quad \cancel{X}\xi_2 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}, \quad \cancel{X}\xi_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$e^{At} = \xi e^{Dt} \xi^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ -79 & 5 & 1 \\ 25 & -1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{3t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ -79 & 5 & 1 \\ 25 & -1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ -79 & 5 & 1 \\ 25 & -1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{3t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{9}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{23}{6} & -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 & 0 \\ -\frac{79}{3}e^{2t} + \frac{45}{2}e^{3t} + \frac{23}{6}e^{-t} & \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t} & \frac{5}{4}e^{3t} - \frac{5}{4}e^{-t} \\ \frac{25}{3}e^{2t} - \frac{9}{2}e^{3t} - \frac{23}{6}e^{-t} & -\frac{1}{4}e^{3t} + \frac{1}{4}e^{-t} & -\frac{1}{4}e^{3t} + \frac{5}{4}e^{-t} \end{bmatrix}$$

$$X(t) = e^{At}X(0) = e^{At}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{2t} \\ -\frac{79}{3}c_1 e^{2t} + \left(\frac{45}{2}c_1 + \frac{5}{4}c_2 + \frac{5}{4}c_3\right)e^{3t} + \left(\frac{23}{6}c_1 - \frac{1}{4}c_2 - \frac{5}{4}c_3\right)e^{-t} \\ \frac{25}{3}c_1 e^{2t} - \left(\frac{9}{2}c_1 + \frac{1}{4}c_2 + \frac{1}{4}c_3\right)e^{3t} + \left(-\frac{23}{6}c_1 + \frac{1}{4}c_1 + \frac{5}{4}c_3\right)e^{-t} \end{bmatrix}$$

Problem 4. (25 points) Determine whether the system described by the following state space equation is: 1) controllable, and 2) observable? Provide your reasons.

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Solution: 1). Controllability:

$$P = [B \; AB \; A^2B \; A^3B \; A^4B \; A^5B \; A^6B]$$

$$= \begin{bmatrix} 1 & -1 & 2 & -1 & 4 & 0 & 8 & 4 & 16 & 16 & 32 & 48 & 64 & 128 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 16 & 0 & 32 & 0 & 64 \\ 1 & 0 & -3 & 0 & 9 & 0 & -27 & 0 & 81 & 0 & -243 & 0 & 729 & 0 \\ 1 & 1 & -3 & -3 & 9 & 9 & -27 & -27 & 81 & 81 & -243 & -243 & 729 & -729 \\ -1 & 1 & -2 & 2 & -4 & 4 & -8 & 8 & -16 & 16 & -32 & 32 & -64 & 64 \\ 0 & 0 & 1 & 0 & 8 & 0 & 48 & 0 & 256 & 0 & 1280 & 0 & 6144 & 0 \\ 1 & 0 & 4 & 0 & 16 & 0 & 64 & 0 & 256 & 0 & 1024 & 0 & 4096 & 0 \end{bmatrix}$$

rank (P) = 7 (full rank) (by finding any 7 linearly independent columns)

$\Rightarrow$ The system is controllable.

2). Observability:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & -1 & +1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 3 & 2 & 0 & 4 \\ 1 & 0 & 3 & 3 & -2 & 4 & 1 \\ & 4 & -9 & -9 & & 0 & 16 \\ 1 & 0 & -9 & -9 & -4 & 16 & 8 \\ 1 & 8 & 27 & 27 & 8 & 0 & 64 \\ & 0 & 27 & 27 & -8 & 64 & 48 \\ 0 & 16 & -81 & -81 & 16 & 0 & 256 \end{bmatrix}$$

Since Q has a column of 0s, rank (Q) ≠ 7, the system is not observable.

Tips :

①. Matrix A is block diagonal.

②. You only need to compute the first 7 column of P matrix and check if they're linearly independent.

③. As long as you identify matrix Q has a complete column of 0s, you know it's not observable.