Week 11 Homework Solutions

Problem 1. (20 points) Consider the following system,

$$\frac{dx}{dt} = -7x + 2y, \frac{dy}{dt} = -6x$$

Find the solution $\xi(t) = \begin{bmatrix} x \\ y \end{bmatrix}$ for the given initial condition $\xi(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solutions: $\dot{\xi} = \begin{bmatrix} -7 & 2 \\ -6 & 0 \end{bmatrix} \xi$ $\qquad A = \begin{bmatrix} -7 & 2 \\ -6 & 0 \end{bmatrix}$

Eigenvalues of A: $\det(\lambda I - A) = 0 \Rightarrow \lambda = -3, -4.$

Eigenvectors: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

$$e^{At} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -3e^{-3t} + 4e^{-4t} & 2e^{-3t} - 2e^{-4t} \\ -6e^{-3t} + 6e^{-4t} & 4e^{-3t} - 3e^{-4t} \end{bmatrix}$$

$$\xi(t) = e^{At}\xi(0) = \begin{bmatrix} -e^{-3t} + 2e^{-4t} \\ -2e^{-3t} + 3e^{-4t} \end{bmatrix}.$$

Problem 2: Given the state space description of the system in terms of x_i (i=1,2) below,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ -21 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

a) Change the state variables and write new state equations for the new states: $\xi_1 = 3x_1 + 2x_2, \xi_2 = 7x_1 + 5x_2$.

b) Is the system controllable?

Solution: a). From $\xi_1(t) = 3x_1(t) + 2x_2(t)$ and $\xi_1(t) = 7x_1(t) + 5x_2(t)$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 11 & 10 \\ -21 & -18 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + u(t)$$

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u(t)$$

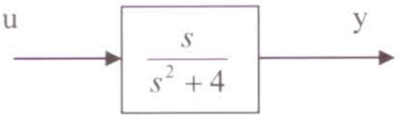
$$y(t) = \begin{bmatrix} -4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + u(t)$$

b). $P = [B \quad AB] = \begin{bmatrix} 0 & 10 \\ 1 & -18 \end{bmatrix}$

rank P = ~~rank~~ 2

So the system is controllable.

Problem 3. (25 points) Consider the system,



a) Write the differential equation governing the dynamics (Note that the transfer function of the system is given in the block diagram);
b) Write its state-space representation;
c) Is the system controllable?

Solution: a) $Y(s) = U(s)\,G(s) = \dfrac{s}{s^2+4}\,U(s)$

$$\Rightarrow \quad s^2 Y(s) + 4Y(s) = s\,U(s) \quad \cdots \text{①}$$

Taking inverse Laplace transform, we obtain

$$\ddot{y}(t) + 4y(t) = \dot{u}(t)$$

b). From equation ①, we get $Y(s) = \frac{1}{s}U(s) - \frac{4}{s^2}Y(s) \triangleq X_1 \quad \cdots \text{①}$

that is, we define $X_1 = \frac{1}{s}U(s) - \frac{4}{s^2}Y(s)$.

We have $sX_1 = U(s) - \frac{4}{s}Y(s) \triangleq U(s) + X_2, \quad \cdots \text{②}$

that is, we define $X_2 = -\frac{4}{s}Y(s)$.

We have $sX_2 = -4\,Y(s) = -4X_1. \quad \cdots \text{③}$

From equation ②, ③, take inverse Laplace transform,

$$\dot{x}_1(t) = x_2(t) + u(t),$$
$$\dot{x}_2(t) = -4x_1(t).$$

From equation ①, $y(t) = x_1(t)$.

So state space representation is $\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

c). $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Controllability matrix :

$$P = [B \quad AB]$$

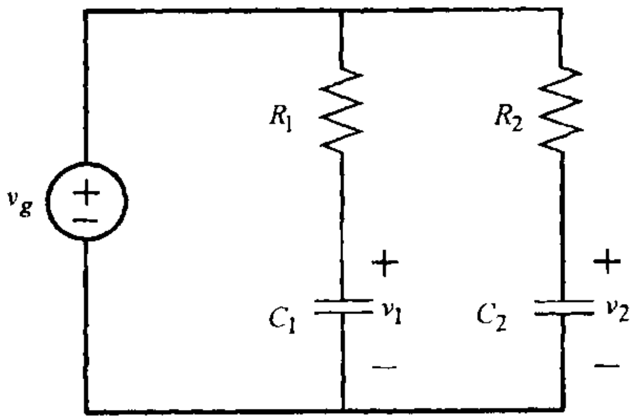
$$= \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$$

$\text{rank}(P) = 2$.

The system is controllable.

Problem 4:

For the circuit shown, find conditions on the system components R_1, R_2, C_1, and C_2 that result in an uncontrollable system. Consider v_g to be the input, and v_1 and v_2 to be the state variables.



Solution: $R_1C_1 = R_2C_2$

Using the KVL,

$$v_g = R_1C_1\frac{dv_1}{dt} + v_1 \qquad\qquad v_g = R_2C_2\frac{dv_2}{dt} + v_2$$

$$\dot{v}_1 = -\frac{1}{R_1C_1}v_1 + \frac{1}{R_1C_1}v_g \qquad\qquad \dot{v}_2 = -\frac{1}{R_2C_2}v_2 + \frac{1}{R_2C_2}v_g$$

$$\dot{v} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} v + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} v_g$$

The state space equation is in the JORDAN form, and obviously, if $\underline{R_1C_1 = R_2C_2}$, the system is UNCONTROLLABLE.