Midterm Exam

EE 602 **Analytic Methods** (Spring **2008)**

Name: ____________________

SID: ____________________

Do all work in the spaces provided. Show all work and organize it for partial credit. Closed-book, closed-notes. 1-page note allowed.

Problem 1. (25 points) Find the eigenvalues and eigenvectors of the matrix A. Is it diagonalizable? If so, find one diagonal matrix.

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}.$$

Solution: $\det(\lambda I - A) = 0$

$$\Rightarrow \det \begin{bmatrix} \lambda-5 & -7 \\ 2 & \lambda+4 \end{bmatrix} = 0 \Rightarrow (\lambda-5)(\lambda+4)+14=0$$

$$\Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow (\lambda-3)(\lambda+2)=0$$

$$\Rightarrow \lambda = 3, -2. \text{ (eigenvalues)}$$

For $\lambda_1 = 3$, $\lambda X_1 = AX_1 \Rightarrow (\lambda I - A) X_1 = 0$

$$\Rightarrow \begin{bmatrix} -2 & -7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{7} \end{bmatrix}.$$

For $\lambda_2 = -2$, $(\lambda_2 I - A) X_2 = 0$

$$\Rightarrow \begin{bmatrix} -7 & -7 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} X_{21} \\ X_{22} \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} X_{21} \\ X_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Diagonal matrix $\Lambda = S^{-1}AS = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$,

and $$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{7} & -1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{7} & -1 \end{bmatrix}.$$

Problem 2. (25 points) For the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

a) **What is the rank of A;**
b) **Find a basis of the row space** R(A) **(Hint: The row space consists of all linear combinations of the rows).**

Solution: a).

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & -2 & 6 & -9 & 1 \\ 0 & -1 & 3 & -4 & 4 \\ 0 & -2 & 6 & -9 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & -2 & 6 & -9 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 3\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{ran}(A) = 3$.

b). $[1 \; 2 \; 1 \; 3 \; 2]$, $[0 \; -2 \; 6 \; -9 \; 1]$,

$[0 \; 0 \; 0 \; \frac{1}{2} \; 3\frac{1}{2}]$.

Problem 3. (25 points) Find the LU decomposition of A:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}.$$

Solution:

$$E_1 A = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ -2 & & 1 & \\ -1 & & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & -2 & 4 \end{bmatrix} \triangleq A_1$$

$$E_2 A_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & -2 & & 1 \end{bmatrix} A_1 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \triangleq A_2$$

So

$$E_2 E_1 A = A_2$$

$$A = (E_2 E_1)^{-1} A_2$$

$$= E_1^{-1} E_2^{-1} A_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\triangleq LU$$

Problem 4. (25 points) Is there a solution to Ax=b for the following two cases of b? If so, find the complete solution and express it as the sum of a particular and a homogeneous solution.

$$A=\begin{bmatrix}1&3&1\\1&2&3\\2&4&6\\1&1&5\end{bmatrix}, \qquad \text{and} \qquad b=\begin{bmatrix}1\\3\\6\\5\end{bmatrix}, \qquad \text{and then} \qquad b=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

Case 1:

Solution: $\operatorname{rank}[A\ \ b] = \operatorname{rank}\begin{bmatrix}1&3&1&1\\1&2&3&3\\2&4&6&6\\1&1&5&5\end{bmatrix} = 2 = \operatorname{rank}(A)$

(Since

$$\bar{A}=\left[\begin{array}{ccc|c}1&3&1&1\\1&2&3&3\\2&4&6&6\\1&1&5&5\end{array}\right] \xrightarrow{E_1\bar{A}} \left[\begin{array}{ccc|c}1&3&1&1\\0&-1&2&2\\0&-2&4&4\\0&-2&4&4\end{array}\right] \xrightarrow{E_2\bar{\bar{A}}} \left[\begin{array}{ccc|c}1&3&1&1\\0&-1&2&2\\0&0&0&0\\0&0&0&0\end{array}\right])$$

there's a solution to $Ax=b$.

We have $E_2E_1AX = E_2E_1b$

Since $E_3\begin{bmatrix}1&3&1\\0&-1&2\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&3&1\\0&-1&2\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&3&1\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix} \triangleq A_3$

$E_4A_3 = \begin{bmatrix}1&-3&&\\&1&&\\&&1&\\&&&1\end{bmatrix}A_3 = \begin{bmatrix}1&0&7\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix} \triangleq A_4$

we have $E_4E_3E_2E_1AX = E_4E_3E_2E_1b$

$$\Rightarrow A_4 X = E_4 E_3 E_2 E_1 b = \begin{pmatrix} 7 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -2 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \end{array}\right] = \left[\begin{array}{c} 7 \\ -2 \\ \hline 0 \\ 0 \end{array}\right]$$

$$I \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 7 \\ -2 \end{bmatrix} x_3 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \end{bmatrix} x_3$$

So $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} k$$

Case 2: Since

$$[A \ b] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -2 & 4 & -2 \\ 0 & -2 & 4 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{rank}\,[A\ b] = 3 \neq \text{rank}(A)$, So ~~there~~'s no solution.