2 (Recommended) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $Ux = 0$ is $x = 0$. Then U has independent columns.

5 Decide the dependence or independence of

(a) the vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1)

(b) the vectors (1, −3, 2) and (2, 1, −3) and (−3, 2, 1).

6 Choose three independent columns of U. Then make two other choices. Do the same for A.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7 If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$ and $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero.

8 If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$ and $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1v_1 + c_2v_2 + c_3v_3 = 0$ in terms of the w's. Find and solve equations for the c's.)

9 Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because ____.

(b) The two vectors v_1 and v_2 will be dependent if ____.

(c) The vectors v_1 and (0, 0, 0) are dependent because ____.

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–15 are about the ... combinations of the vectors.

11 Describe the subspace of $\mathbf{R}$...

(a) the two vectors (1, 1, ...

(b) the three vectors (0, ...

(c) the columns of a 3 by ...

(d) all vectors with posi...

12 The vector b is in the subsp... solution to ____. The vect... to ____.

True or false: If the zero ve...

13 Find the dimensions of the... (a) column space of A, (b)... space of U:

$$A = \begin{bmatrix} 1 & \cdots \\ 1 & \cdots \\ 3 & \cdots \end{bmatrix}$$

14 Choose $x = (x_1, x_2, x_3, x_4)$ ... and (x_4, x_3, x_1, x_2). Those 24 ... specific vectors x so that the ... (d) four.

15 $v + w$ and $v - w$ are combina... of $v + w$ and $v - w$. The two ... they a basis for the same spa...

Questions 16–26 are about the re...

16 If $v_1, \ldots, v_n$ are linearly indep... These vectors are a ____ for ... m by n matrix, then m is ____

17 Find a basis for each of these ...

(a) All vectors whose compo...

(b) All vectors whose compo...

(c) All vectors that are perpe...

(d) The column space (in $\mathbf{R}$...

26 For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Questions 27–32 are about spaces where the "vectors" are matrices.

27 Find a basis for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All symmetric matrices ($A^{\mathrm{T}} = A$).

(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

28 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$.

29 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

30 Show that the six 3 by 3 permutation matrices (Section 2.6) are linearly dependent.

31 What subspace of 3 by 3 matrices is spanned by

(a) all invertible matrices?

(b) all echelon matrices?

(c) the identity matrix?

32 Find a basis for the space of 2 by 3 matrices whose nullspace contains (2, 1, 1).

Questions 33–37 are about spaces where the "vectors" are functions.

33 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

34 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace with $y(0) = 0$.

35 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$

(b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

36 Suppose $y_1(x), y_2(x), y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

37 Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

38 Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a+c+d = 0$ and also for the space $\mathbf{T}$ with $a+b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

39 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

40 If $AS = SA$ for the *shift matrix* S, show that A must have this special form:

$$\text{If} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

"The subspace of matrices that commute with the shift S has dimension ____."

41 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$

(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$

(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$

(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$

42 Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[\,A \;\; b\,]$ is invertible. Show that $Ax = b$ is solvable when $[\,A \;\; b\,]$ is singular.

DIMEN

The main theorem in this chapter ... is the number of pivots. The dim... basis. We count pivots or we count ... *of all four fundamental subspaces* ...

Two subspaces come directly ...

Four Fundamental Subspaces

1. The ***row space*** is $C(A^{\mathrm{T}})$, a ...
2. The ***column space*** is $C(A)$, a ...
3. The ***nullspace*** is $N(A)$, a sub...
4. The ***left nullspace*** is $N(A^{\mathrm{T}})$, ...

In this book the column space and ... pretty well. Now the other two sub... combinations of the rows. *This is the* ...

For the left nullspace we solve ... *nullspace of* A^{T}. The vectors y go on ... as $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$. The matrices A and A^{T} ... and their nullspaces. But those spaces ...

Part 1 of the Fundamental Theorem ... One fact stands out: ***The row space and*** ... (the rank of the matrix). The other im... *dimensions are* $n - r$ *and* $m - r$, to ...

Part 2 of the Fundamental Theorem ... gether (two in $\mathbf{R}^n$ and two in $\mathbf{R}^m$). Th... $Ax = b$. Stay with it—you are doing ...

Suppose A is reduced to its row echelon ... spaces are easy to identify. We will find ... mension. Then we watch how the subspaces ... back at A. The main point is that *the fo*...

As a specific 3 by 5 example, look at ...

$$m = 3 \quad n = 5 \quad r = 2 \qquad \begin{bmatrix} 1 & 3 & 5 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$