Finding e^A where $A = 2I_3 - J_3$

HW 8

From the $aI_n + bJ_n$ notes we get that the eigenvalues are 2, 2, −1

The matrix $M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has as its 3 columns eigenvectors

corresponding to eigenvalues 2, 2, −1, i.e.,

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{Now} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\det M}\begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}^t = \frac{1}{\det M}\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\& \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\right) = 3$$

$$\text{So} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \text{diag}(2,2,-1)$$

So $M^{-1}AM = \text{diag}(2,2,-1)$, & this same M will diagonalize all powers of A.

So $M^{-1}e^AM = \text{diag}(e^2, e^2, e^{-1})$, and $e^A = M\,\text{diag}(e^2, e^2, e^{-1})M^{-1}$ or

$$e^A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\text{diag}(e^2, e^2, e^{-1})\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} -e^2 & -e^2 & e^{-1} \\ e^2 & 0 & e^{-1} \\ 0 & e^2 & e^{-1} \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} (2/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} \\ -(1/3)e^2 + (1/3)e^{-1} & (2/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} \\ -(1/3)e^2 + (1/3)e^{-1} & -(1/3)e^2 + (1/3)e^{-1} & (2/3)e^2 + (1/3)e^{-1} \end{bmatrix}$$

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Which we will check using MATLAB.
A=[1,-1,-1;-1,1,-1;-1,-1,1]


   1  -1  -1
  -1   1  -1
  -1  -1   1

>> B=expm(A)

B =

   5.0487  -2.3404  -2.3404
  -2.3404   5.0487  -2.3404
  -2.3404  -2.3404   5.0487
```