# Exam #1

## EEL 6621 Nonlinear Control (Fall 2004)

**Name:** Solutions

**SS#:** ______

Lecture notes and one textbook are allowed. Please show all work for partial credit (100 points total). If the space provided is not enough, please ask for extra blank sheets of paper.

**Problem 1.** (40 points) Consider the following nonlinear system

$$\dot{x}_1 = -3x_1 + x_1^3 - x_2,$$
$$\dot{x}_2 = x_1 - ax_2,$$

(a) Determine the local stability properties of all equilibrium points to the nonlinear system if a=1;

(b) Let a=0. Prove that every solution starting close to the origin will approach the origin. You may need to use a quadratic Lyapunov function and the invariant set theorem.

Solution. a). Equilibrium: $\begin{cases} -3x_1 + x_1^3 - x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$

$\Rightarrow \quad x_1^* = x_2^* = 0, \ \pm 2.$ So: $(0,0)$, $(2,2)$, $(-2,-2)$ are equilibrium points.

Linearization around equilibrium $(x_1^*, x_2^*)$ gives:

$$A(x_1^*, x_2^*) = \begin{bmatrix} -3+3x_1^2 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A(0,0) = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$$ eigenvalues: $-2, -2$. locally stable

$$A(-2,-2) = A(2,2) = \begin{bmatrix} 9 & -1 \\ 1 & -1 \end{bmatrix}$$ eigenvalues: $4 \pm \sqrt{24}$. locally unstable.

b).

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \qquad \text{p.d.}$$

$$\dot{V}(x) = x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2$$

$$= -3x_1^2 + x_1^4 - x_1 x_2 + x_1 x_2$$

$$= x_1^2(x_1^2 - 3) \le 0 \quad \text{when} \quad |x_1| < \sqrt{3}.$$

locally n.s.d.

$$\dot{V}(x) = 0 \quad \Rightarrow \quad x_1 = 0$$

$$\Rightarrow \dot{x}_1 = 0, \quad \text{from} \quad \dot{x}_1 = -3x_1 + x_1^3 - x_2$$

$$\Rightarrow x_2 = 0$$

So in a neighborhood of the origin $R: \{|x_1| < \sqrt{3}, |x_2| < \text{[illegible]}\}$,

$\dot{V}(x) = 0$ contains no traj. other than

$$(x_1, x_2) = (0, 0).$$

From the invariant set theorem,

$(x_1^*, x_2^*) = (0,0)$ is locally a.s.

**Problem 2.** (30 points) Use a quadratic Lyapunov function to show that the origin is exponentially stable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad |\alpha(t)| \le 1.$$

Solution: Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$$\dot{V}(x) = x_1 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2$$

$$= -x_1^2 + 2\alpha(t)\, x_1 x_2 - 2x_2^2$$

$$\le -x_1^2 + 2|x_1| \cdot |x_2| - 2x_2^2$$

Due to $2\alpha(t) x_1 x_2 \le 2|\alpha(t)| \cdot |x_1| \cdot |x_2| \le 2|x_1| \cdot |x_2|$.

So $\dot{V}(x) \le -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$

eigenvalues of $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ are $\frac{3 \pm \sqrt{5}}{2}$;

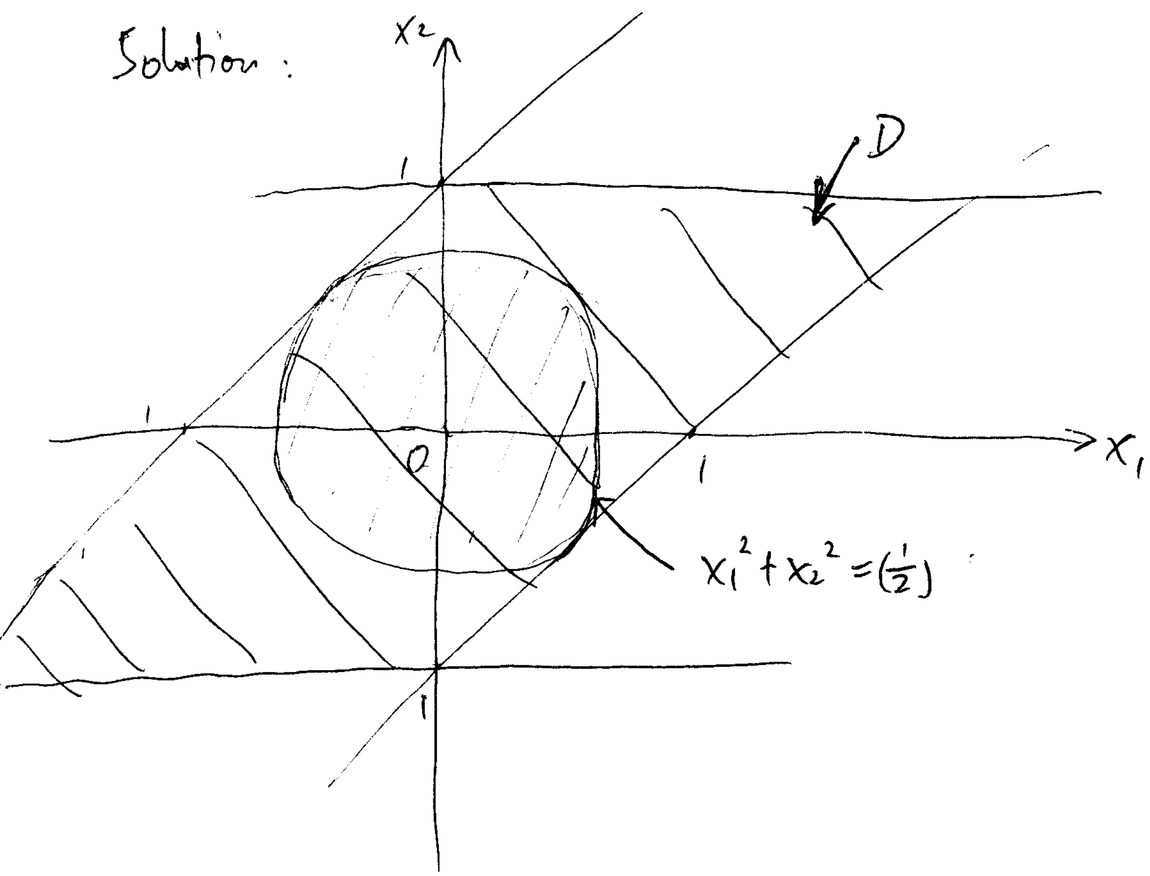
since $\lambda_{min}(P)|x|^2 \le x^T P x \le \lambda_{max}(P)|x|^2$,

$$\dot{V}(x) \le -0.382\,(x_1^2 + x_2^2)$$

Therefore, the equilibrium is exponentially stable.

**Problem 3.** (30 points) Consider a second-order system $\dot{x} = f(x)$ with asymptotically stable origin. Let $V(x) = x_1^2 + x_2^2$, and $D = \{x \in R^2 \mid |x_2| < 1, |x_1 - x_2| < 1\}$. Suppose $[\partial V / \partial x] f(x)$ is negative definite in D. Estimate the region of attraction.

Solution:




The biggest region defined by $R: \{ V(x) = x_1^2 + x_2^2 = C \}$ in D is $C = \frac{1}{2}$, (i.e., $C = \min_{x \in D} V(x)$).

So the region of attraction is $\{ (x_1, x_2) \mid x_1^2 + x_2^2 < (\frac{1}{2}) \}$.