- **1.9** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2\\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C} i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right]\\ &= \frac{1}{C}\left[i_s - I_0 \sin k x_1 - \frac{1}{R} x_2\right]\end{aligned}$$

(b) Let $x_1 = i_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= I_0 k \cos k\phi_L \; \dot{\phi}_L = k\sqrt{I_0^2 - i_L^2}\, v_C \;.\\ &= x_2 k \sqrt{I_0^2 - x_1^2}\\ \dot{x}_2 &= \frac{1}{C}\left[i_s - x_1 - \frac{1}{R} x_2\right]\end{aligned}$$

The model of (a) is more familiar since it is the pendulum equation.

- **1.10** **(a)** Let $x_1 = \phi_L$, $x_2 = v_C$.

$$\begin{aligned}\dot{x}_1 &= \dot{\phi}_L = v_L = v_C = x_2\\ \dot{x}_2 &= \dot{v}_C = \frac{1}{C} i_C = \frac{1}{C}\left[i_s - \frac{v_C}{R} - i_L\right]\\ &= \frac{1}{C}\left[i_s - L x_1 - \mu x_1^3 - \frac{1}{R} x_2\right]\end{aligned}$$

(b) $x_2 = 0 \;\Rightarrow\; L x_1 + \mu x_1^3 = 0 \;\Rightarrow\; x_1 = 0$. There is a unique equilibrium point at the origin.

- **1.12** The equation of motion is

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Let $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{k}{M} x_1 - \frac{c_1}{M} x_2 - \frac{c_2}{M} x_2 |x_2| + g$$

- **1.13** **(a)**

$$m\ddot{y} = -(k_1 + k_2)y - c\dot{y} + h(v_0 - \dot{y})$$

where $c > 0$ is the viscous friction coefficient.
(b) $h(v) \approx h(v_0) - h'(v_0)\dot{y}$.

$$m\ddot{y} = -(k_1 + k_2)y - [c + h'(v_0)]\dot{y} + h(v_0)$$

(c) To obtain negative friction, we want $c + h'(v_0) < 0$. This can be achieved with the friction characteristic of Figure 1.5(d) if v_0 is in the range where the slope is negative and the magnitude of the negative slope is greater than c.

- **1.15** **(a)**

$$H = m\frac{d^2}{dt^2}(y + L\sin\theta) = m\frac{d}{dt}(\dot{y} + L\dot{\theta}\cos\theta) = m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$V = m\frac{d^2}{dt^2}(L\cos\theta) + mg = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) + mg = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + mg$$

Substituting V and H in the $\ddot{\theta}$-equation yields

$$\begin{aligned} I\ddot{\theta} &= VL\sin\theta - HL\cos\theta \\ &= -mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta + mgL\sin\theta \\ &\quad -mL\ddot{y}\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &= -mL^2\ddot{\theta}[(\sin\theta)^2 + (\cos\theta)^2] + mgL\sin\theta - mL\ddot{y}\cos\theta \\ &= -mL^2\ddot{\theta} + mgL\sin\theta - mL\ddot{y}\cos\theta \end{aligned}$$

Substituting H in the $\ddot{y}$-equation yields

$$M\ddot{y} = F - m(\ddot{y} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta) - k\dot{y}$$

(b)

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}$$

$$\det(D(\theta)) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta = \Delta(\theta)$$

Hence,

$$D^{-1}(\theta) = \frac{1}{\Delta(\theta)}\begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)}\begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix}\begin{bmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{bmatrix}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\theta} = \frac{1}{\Delta(\theta)}\left[(m + M)mgL\sin\theta - mL\cos\theta(F + mL\dot{\theta}^2\sin\theta - k\dot{y})\right] \\ &= \frac{1}{\Delta(x_1)}\left[(m + M)mgL\sin x_1 - mL\cos x_1(u + mLx_2^2\sin x_1 - kx_4)\right] \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddot{y} = \frac{1}{\Delta(\theta)}\left[-m^2L^2g\sin\theta\cos\theta + (I + mL^2)(F + mL\dot{\theta}^2\sin\theta - k\dot{y})\right] \\ &= \frac{1}{\Delta(x_1)}\left[-m^2L^2g\sin x_1\cos x_1 + (I + mL^2)(u + mLx_2^2\sin x_1 - kx_4)\right] \end{aligned}$$

- **1.16 (a)**

$$F_x = m\frac{d^2}{dt^2}(x_c + L\sin\theta) = m\frac{d}{dt}(\dot{x}_c + L\dot{\theta}\cos\theta) = m(\ddot{x}_c + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)$$

$$F_y = m\frac{d^2}{dt^2}(L\cos\theta) = m\frac{d}{dt}(-L\dot{\theta}\sin\theta) = -mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta$$

Substituting F_x and F_y in the $\ddot{\theta}$-equation yields

$$\begin{aligned} I\ddot{\theta} &= u + F_y L\sin\theta - F_x L\cos\theta \\ &= u - mL^2\ddot{\theta}(\sin\theta)^2 - mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &\quad -mL\ddot{x}_c\cos\theta - mL^2\ddot{\theta}(\cos\theta)^2 + mL^2\dot{\theta}^2\sin\theta\cos\theta \\ &= u - mL^2\ddot{\theta}[(\sin\theta)^2 + (\cos\theta)^2] - mL\ddot{x}_c\cos\theta \\ &= u - mL^2\ddot{\theta} - mL\ddot{x}_c\cos\theta \end{aligned}$$

Substituting F_x in the $\ddot{x}_c$-equation yields

$$M\ddot{x}_c = -m\ddot{x}_c - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta - kx_c$$

Thus,

$$D(\theta)\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

where

$$D(\theta) = \begin{bmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & m + M \end{bmatrix}$$

(b)

$$\det(D(\theta)) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta = \Delta(\theta)$$

Hence,

$$D^{-1}(\theta) = \frac{1}{\Delta(\theta)}\begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x}_c \end{bmatrix} = \frac{1}{\Delta(\theta)}\begin{bmatrix} m + M & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{bmatrix}\begin{bmatrix} u \\ mL\dot{\theta}^2\sin\theta - kx_c \end{bmatrix}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\theta} = \frac{1}{\Delta(\theta)}\left[(m + M)u - mL\cos\theta(mL\dot{\theta}^2\sin\theta - kx_c)\right] \\ &= \frac{1}{\Delta(x_1)}\left[(m + M)u - mL\cos x_1(mLx_2^2\sin x_1 - kx_3)\right]. \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddot{x}_c = \frac{1}{\Delta(\theta)}\left[-mLu\cos\theta + (I + mL^2)(mL\dot{\theta}^2\sin\theta - kx_c)\right] \\ &= \frac{1}{\Delta(x_1)}\left[-mLu\cos x_1 + (I + mL^2)(mLx_2^2\sin x_1 - kx_3)\right] \end{aligned}$$

(d) Take $u =$ constant. Setting the derivatives $\dot{x}_i = 0$, we obtain $x_2 = x_4 = 0$ and

$$\begin{aligned} 0 &= (m + M)u + mkLx_3\cos x_1 \\ 0 &= -mLu\cos x_1 - k(I + mL^2)x_3 \end{aligned}$$

Eliminating x_3 between the two equations yields

$$u[(m + M)(I + mL^2) - m^2L^2\cos^2 x_1] = u\,\Delta(x_1) = 0$$

Since $\Delta(x_1) > 0$, equilibrium can be maintained only at $u = 0$. Then, $x_3 = 0$. Thus, the system has an equilibrium set $\{x_2 = x_3 = x_4 = 0\}$.