• **2.2 (1)**

$$0 = x_2, \quad 0 = -x_1 + (1/16)x_1^5 - x_2$$

$$x_2 = 0 \;\Rightarrow\; 0 = x_1(x_1^4 - 16) \;\Rightarrow\; x_1 = 0,\ 2,\ \text{or}\ -2$$

There are three equilibrium points at $(0,0)$, $(2,0)$, and $(-2,0)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + (5/16)x_1^4 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{3}/2 \;\Rightarrow\; (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,0)} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm \sqrt{17}/2 \;\Rightarrow\; (2,0) \text{ is a saddle}$$

Similarly, $(-2,0)$ is a saddle.

(2)

$$0 = 2x_1 - x_1x_2, \quad 0 = 2x_1^2 - x_2$$

$$x_1(2 - x_2) = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_2 = 2$$

$$x_1 = 0 \;\Rightarrow\; x_2 = 0, \quad x_2 = 2 \;\Rightarrow\; x_1^2 = 1 \;\Rightarrow\; x_1 = 1 \text{ or } -1$$

There are three equilibrium points at $(0,0)$, $(1,2)$, and $(-1,2)$. Determine the type of each point using linearization.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = 2, -1 \;\Rightarrow\; (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (1,2) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \;\Rightarrow\; \lambda_{1,2} = -(1/2) \pm j\sqrt{15}/2 \;\Rightarrow\; (-1,2) \text{ is a stable focus}$$

(3)

$$0 = x_2, \quad 0 = -x_2 - \psi(x_1 - x_2)$$

$$x_2 = 0 \;\Rightarrow\; \psi(x_1) = 0 \;\Rightarrow\; x_1 = 0$$

There is a unique equilibrium point at $(0,0)$. Determine its type by linearization.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -3(x_1 - x_2)^2 - 0.5 & -1 + 3(x_1 - x_2)^2 + 0.5 \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}$$

The eigenvalues are $-(1/4) \pm j\sqrt{7}/4$. Hence, $(0,0)$ is stable focus.

• **2.4** **(1)** The system has three equilibrium points at $(0, 0)$, $(a, 0)$, and $(-a, 0)$, where a is the root of

$$a = \tan(a/2) \quad \Rightarrow \quad a \approx 2.3311$$

The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - 2/[1 + (x_1 + x_2)^2] & -2/[1 + (x_1 + x_2)^2] \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda_{1,2} = -1, \; -1$$

Although we have multiple eigenvalues, we can conclude that the origin is a stable node because $f(x)$ is an analytic function of x in the neighborhood of the origin.

$$\left.\frac{\partial f}{\partial x}\right|_{(2.3311,0)} = \begin{bmatrix} 0 & 1 \\ 0.6892 & -0.3108 \end{bmatrix} \quad \Rightarrow \quad \lambda_{1,2} = 0.6892, \; -1 \quad \Rightarrow \quad (2.3311, 0) \text{ is a saddle}$$

Similarly, $(-2.3311, 0)$ is a saddle. The phase portrait is shown in Figure 2.7 with the arrowheads. The stable trajectories of the two saddle points forms two separatrices, which divide the plane into three regions. All trajectories in the middle region converge to the origin as t tends to infinity. All trajectories in the outer