Chapter 3

• **3.1** **(1)** The term $|x|$ is not continuously differentiable at $x = 0$, but it is globally Lipschitz. The term x^2 is continuously differentiable, but its partial derivative is not globally bounded. Thus $f = x^2 + |x|$ is not continuously differentiable at $x = 0$. It is continuously differentiable on a domain that does not include $x = 0$. It is locally Lipschitz, hence continuous, but not globally Lipschitz.
(2) The term $\text{sgn}(x)$ is discontinuous at $x = 0$. Thus, $f(x) = x + \text{sgn}(x)$ does not have any of the four properties in a domain that contains $x = 0$.
(3) $f(x) = \sin(x)\ \text{sgn}(x)$ is globally Lipschitz. This can be seen as follows. If both x and y are nonnegative, we have

$$|f(x) - f(y)| = |\sin(x) - \sin(y)| \leq |x - y|$$

If $x \geq 0$ and $y \leq 0$, we have

$$|f(x) - f(y)| = |\sin(x) + \sin(y)| = |2\sin(\tfrac{1}{2}(x + y))\cos(\tfrac{1}{2}(x - y))| \leq |x - y|$$

Other cases can be dealt with similarly to conclude that $|f(x) - f(y)| \leq |x - y|$ for all x, y. It follows that f is both locally Lipschitz and continuous. It is not continuously differentiable at $x = 0$ because $\lim_{x\to 0+} f'(x) = +1$ while $\lim_{x\to 0-} f'(x) = -1$.
(4) $f(x) = -x + a\sin x$ is continuously differentiable. Hence, it is locally Lipschitz and continuous. $\frac{df}{dx} = -1 + a\cos x$ is globally bounded. Hence, it is globally Lipschitz.
(5) $f(x) = -x + 2|x|$ is not continuously differentiable. It is globally Lipschitz because both x and $|x|$ are so. Hence, it is locally Lipschitz.
(6) $f(x) = \tan(x)$ is continuously differentiable in the open interval $-\pi/2 < x < \pi/2$. Hence, it is locally Lipschitz and continuous in the same interval. Its derivative $\sec^2(x)$ is not globally bounded; hence, it is not globally Lipschitz.
(7) The function $\tanh(y)$ is continuously differentiable and its derivative $1/cosh^2(y)$ is globally bounded; hence it is globally Lipschitz. Clearly, the linear function y is both continuously differentiable and globally Lipschitz. Hence, f has all four properties.
(8) f is not continuously differentiable due to the term $|x_2|$ in f_1. Check the Lipschitz property component by component. f_1 is globally Lipschitz as can be easily checked. f_2 is continuously differentiable, but its partial derivatives are not globally bounded. Hence f_2 is locally Lipschitz but not globally so. Since both f_1 and f_2 are locally Lipschitz, so is f. Since f is locally Lipschitz, it is continuous. f is not globally Lipschitz since f_2 is not so.

• **3.4** The function f can be written as $f(x) = g(x)Kxh(\psi(x))$ where

$$h(\psi) = \begin{cases} \frac{1}{\psi}, & \text{if } \psi \geq \mu > 0 \\ \frac{1}{\mu}, & \text{if } \psi < \mu \end{cases} \qquad \text{and} \quad \psi(x) = g(x)\|Kx\|$$

The norm function $\|Kx\|$ is Lipschitz since

$$|\ \|Kx\| - \|Ky\|\ | \leq \|Kx - Ky\| \leq \|K\|\ \|x - y\|$$

• **3.6 (a)**

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\ d\tau \\ \|x(t)\| &\leq \|x_0\| + \int_{t_0}^{t} \|f(\tau, x(\tau))\|\ d\tau \\ &\leq \|x_0\| + \int_{t_0}^{t} [k_1 + k_2\|x(\tau)\|]\ d\tau \\ &= \|x_0\| + k_1(t - t_0) + k_2 \int_{t_0}^{t} \|x(\tau)\|\ d\tau \end{aligned}$$

By Gronwall-Bellman inequality

$$\|x(t)\| \leq \|x_0\| + k_1(t - t_0) + \int_{t_0}^{t} [\|x_0\| + k_1(s - t_0)]k_2 e^{k_2(t-s)}\ ds$$

Integrating by parts, we obtain

$$\|x(t)\| \leq \|x_0\| \exp[k_2(t - t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t - t_0)] - 1\}, \quad \forall\, t \geq t_0$$

(b) The upper bound on $\|x(t)\|$ is finite for every finite t. It tends to ∞ as $t \to \infty$. Hence the solution of the system cannot have a finite escape time.