• **4.2** Let $f(x) = ax^p + g(x)$. Near the origin, the term ax^p is dominant. Hence, $\text{sign}(f(x)) = \text{sign}(ax^p)$. Consider the case when $a < 0$ and p is odd. With $V(x) = \frac{1}{2}x^2$ as a Lyapunov function candidate, we have

$$\dot{V} = x[ax^p + g(x)] \le ax^{p+1} + k|x|^{p+2}$$

Near the origin, the term ax^{p+1} is dominant. Hence, $\dot{V}(x)$ is negative definite and the origin is asymptotically stable. Consider now the case when $a > 0$ and p is odd. In the neighborhood of the origin, $\text{sign}(f(x)) = \text{sign}(x)$. Hence, a trajectory starting near $x = 0$ will be always moving away from $x = 0$. This shows that the origin is unstable. When p is even, a similar behavior will take place on one side of the origin; namely, on the side $x > 0$ when $a > 0$ and $x < 0$ when $a < 0$. Therefore, the origin is unstable.

• **4.3** **(1)** Let $V(x) = (1/2)(x_1^2 + x_2^2)$.

$$\dot{V} = x_1(-x_1 + x_1x_2) - x_2^2$$

In the set $\{\|x\|_2 \le r^2\}$, we have $|x_1| \le r$. Hence,

$$\dot{V} \le -x_1^2 - x_2^2 + r|x_1|\,|x_2| = -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -r/2 \\ -r/2 & 1 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$$

$\dot{V}$ is negative definite for $r < 2$. Thus, the origin is asymptotically stable. To investigate global asymptotic stability, note that the solution of the second equation is $x_2(t) = \exp(-t)x_2(0)$, which when substituted in the first equation yields

$$\dot{x}_1 = [-1 + \exp(-t)x_2(0)]x_1$$

This is a linear time-varying system whose solution does not have a finite escape time. After some finite time the coefficient of x_1 on the right-hand side will be less than a negative number. Hence, $\lim_{t\to\infty} x_1(t) = 0$. Thus, the origin is globally asymptotically stable.
(2) Let $V(x) = (1/2)(x_1^2 + x_2^2)$.

$$\dot{V} = -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) = -2V(1 - 2V)$$

In the region $V(x) < 1/2$, $\dot{V}$ is negative definite. Hence, the origin is asymptotically stable. For $V > 1/2$, $\dot{V}$ is positive. Hence, trajectories starting in the region $V(x) > 1/2$ cannot approach the origin. In fact, they grow unbounded. Thus, the origin is not globally asymptotically stable.
(3) Let $V(x) = x^TPx = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$, where P is a positive definite symmetric matrix.

$$\dot{V} = -2p_{12}x_1^2 + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2(p_{22} - p_{12})x_2^2 + \text{Higher order terms}$$

Near the origin, the quadratic term dominates the higher-order terms. Thus, $\dot{V}$ will be negative definite in the neighborhood of the origin if the quadratic term is negative definite. Choosing $p_{12} = 1$, $p_{22} = 2$, and $p_{11} = 3$ makes $V(x)$ positive definite and $\dot{V}(x)$ negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since the origin is not the unique equilibrium point. The set $\{x_1^2 = 1\}$ is an equilibrium set.
(4) Let $V(x) = x_1^2 + (1/2)x_2^2$.

$$\dot{V} = -2x_1^2 - 2x_1x_2 + 2x_1x_2 - x_2^4 = -x_1^2 - x_2^4$$

Hence, the origin is globally asymptotically stable.

• **4.6** Try

$$g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

To meet the symmetry requirement, take $\gamma = \beta$.

$$\dot{V}(x) = (\alpha x_1 + \beta x_2)x_2 - (\beta x_1 + \delta x_2)[(x_1 + x_2) + h(x_1 + x_2)]$$

Take $\delta = \beta$.

$$\dot{V}(x) = -\beta x_1^2 + (\alpha - 2\beta)x_1 x_2 - \beta(x_1 + x_2)h(x_1 + x_2)$$

Taking $\alpha = 2\beta$ and $\beta > 0$ yields

$$\dot{V}(x) = -\beta x_1^2 - \beta(x_1 + x_2)h(x_1 + x_2)$$

which is negative definite for all $x \in R^2$. Now

$$g(x) = \beta \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x \stackrel{\text{def}}{=} Px \;\Rightarrow\; V(x) = \int_0^x g^T(y)\, dy = \frac{1}{2} x^T P x$$

where P is positive definite. Thus, $V(x)$ is a radially unbounded Lyapunov function and the origin is globally asymptotically stable.

• **4.7** **(a)** Let $\nabla V(x) = g(x)$. Then, $\dot{V} = -g^T(x)Q\phi(x)$. Choose $g(x) = Px$ so that $V(x) = (1/2)x^T P x$. We need to choose $P = P^T > 0$ such that $\dot{V} = -x^T PQ\phi(x)$ is negative definite. Choosing $P = Q^{-1}$ yields

$$\dot{V} = -x^T \phi(x) = -\sum_{i=1}^{n} x_i \phi_i(x_i)$$

$\dot{V}$ is negative definite in the neighborhood of the origin because $y\phi(y) > 0$ for $y \neq 0$. Hence, the origin is asymptotically stable.
(b) The function $V(x)$ is radially unbounded. The origin will be globally asymptotically stable if $\dot{V}$ is negative definite for all x. This will be the case if $y\phi_i(y) > 0$ for all $y \neq 0$.
(c) The function ϕ_2 satisfies the condition $y\phi_i(y) > 0$ for all $y \neq 0$. The function ϕ_1 satisfies the condition only near $y = 0$ because $\phi_1(y)$ vanishes at $y = 1$. Thus, we can only show asymptotic stability of the origin using the Lyapunov function $V(x) = x^T Q^{-1} x = x_1^2 + 2x_1x_2 + 2x_2^2$.

• **4.9** **(a)**

$$x_1 = 0 \;\Rightarrow\; V(x) = \frac{x_2^2}{1 + x_2^2} + x_2^2 \;\rightarrow \infty \text{ as } |x_2| \rightarrow \infty$$

$$x_2 = 0 \;\Rightarrow\; V(x) = \frac{x_1^2}{1 + x_1^2} + x_1^2 \;\rightarrow \infty \text{ as } |x_1| \rightarrow \infty$$

(b) On the line $x_2 = x_1$, we have

$$V(x) = \frac{4x_1^2}{1 + 4x_1^2} \;\rightarrow\; 1 \text{ as } |x_1| \rightarrow \infty$$