• **4.14**

$$\int_0^{x_1} yg(y)\ dy \geq \int_0^{x_1} y\ dy = \frac{1}{2}x_1^2$$

Therefore.

$$V(x) \geq \tfrac{1}{2}x_1^2 + x_1x_2 + x_2^2 = \tfrac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x$$

The matrix of the quadratic form is positive definite. Hence, $V(x)$ is positive definite for all x, and radially unbounded.

$$\begin{aligned} \dot{V} &= x_1 g(x_1)\dot{x}_1 + x_1\dot{x}_2 + x_2\dot{x}_1 + 2x_2\dot{x}_2 \\ &= g(x_1)(x_1x_2 - x_1^2 - x_1x_2 - 2x_1x_2 - 2x_2^2) + x_2^2 \\ &= -g(x_1)(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 \\ &= -g(x_1)x^TQx + x_2^2 \end{aligned}$$

where $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is positive definite. Since $g(x_1) \geq 1$ and $x^TQx \geq 0$, we have

$$\dot{V} \leq -(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 = -(x_1^2 + 2x_1x_2 + x_2^2) = -(x_1 + x_2)^2$$

This shows that $\dot{V}$ is negative semidefinite. We need to apply the invariance principle.

$$\dot{V} = 0 \ \Rightarrow \ 0 \leq -(x_1 + x_2)^2 \ \Rightarrow \ 0 \geq (x_1 + x_2)^2 \ \Rightarrow \ x_1 + x_2 = 0$$

$$x_1(t) + x_2(t) \equiv 0 \ \Rightarrow \ \dot{x}_1(t) + \dot{x}_2(t) \equiv 0 \ \Rightarrow \ x_2(t) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$$

Since $V(x)$ is radially unbounded and all the assumptions hold globally, we conclude that the origin is globally asymptotically stable.

• **4.15** **(a)** The equilibrium points are the roots of the equations

$$0 = x_2, \quad 0 = -h_1(x_1) - x_2 - h_2(x_3), \quad 0 = x_2 - x_3$$

$$x_2 = 0 \ \Rightarrow \ x_3 = 0 \ \Rightarrow \ h_1(x_1) = 0 \ \Rightarrow \ x_1 = 0$$

Hence, there is a unique equilibrium point at the origin.

(b) $V(x)$ is the sum of nonnegative terms; hence $V(x) \geq 0$. To show that it is positive definite for all x, we need to show that $V(x) = 0 \ \Rightarrow \ x = 0$. Since $yh_i(y) > 0$ for all $y \neq 0$, the integrals $\int_0^{x_1} h_1(y)\ dy$ and $\int_0^{x_3} h_2(y)\ dy$ vanish only at $x_1 = 0$ and $x_3 = 0$, respectively. Hence, $V(x)$ is positive definite.

(c)

$$\dot{V} = h_1(x)x_2 + x_2[-h_1(x_1) - x_2 - h_2(x_3)] + h_2(x_3)(x_2 - x_3) = -x_2^2 - x_3h_2(x_3)$$

$\dot{V}(x)$ is negative semidefinite for all x, but not negative definite because $\dot{V}(x) = 0$ when $x_2 = x_3 = 0$ for any x_1. We apply the invariance principle.

$$x_2(t) \equiv 0 \text{ and } x_3(t) \equiv 0 \ \Rightarrow h_1(x_1(t)) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$$

Hence, the origin is asymptotically stable.

(d) To show global asymptotic stability we need $V(x)$ to be radially unbounded. This will be the case if the integrals $\int_0^z h_i(y)\ dy$, $i = 1, 2$, tend to infinity as $|z| \to \infty$.

• **4.18** The system has an equilibrium point at $y = Mg/k$ and $\dot{y} = 0$. Let $x_1 = y - Mg/k$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|$$

Take $V(x) = ax_1^2 + bx_2^2$, with $a, b > 0$. $V(x)$ is positive definite and radially unbounded.

$$\dot{V}(x) = 2\left(a - \frac{bk}{M}\right)x_1x_2 - - \frac{2bc_1}{M}x_2^2 - \frac{2bc_2}{M}x_2^2|x_2|$$

Taking $a = k/2$ and $b = M/2$, we obtain

$$\dot{V}(x) = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0, \quad \forall\, x$$

Moreover,

$$\dot{V} \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Using LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.