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Nonlinear Cooperative Control for Consensus of Nonlinear and Heterogeneous Systems

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Abstract— **In this paper, the consensus problem is considered for nonlinear and heterogeneous systems. Topology of their sensing/communication network is allowed to change in an arbitrary and intermittent way. A matrix-theoretical approach is used to reveal the necessary and sufficient condition of cooperative controllability. The condition is then used to search for cooperative control Lyapunov function (with the same Lyapunov function components) for linear cooperative systems. It is shown that, although finding cooperative control Lyapunov function is often too difficult for nonlinear systems, their cooperative stability can be concluded if the Lyapunov function components satisfy certain differential inequalities along system trajectory. This new result enables an explicit Lyapunov argument with respect to topology changes and a constructive procedure of designing nonlinear cooperative controls for a class of nonlinear and heterogeneous systems. Examples are presented to illustrate effectiveness of the proposed nonlinear cooperative controls.**

I. Introduction

Significant progress has been made in cooperative control of dynamical systems over the past several years, and a typical setting is the consensus problem in which dynamical systems are desired to have their outputs reach a common consensus value. A distinctive feature of cooperative systems is that cooperative control of a specific system often has to depend on intermittent feedbacks from those systems in its vicinity. As a result, design and analysis of cooperative control requires certain connectivity conditions on the sensing/communication network among the dynamical systems. Along this line of research, the consensus problem has been investigated, especially for linear dynamical systems. For instance, it is shown in [3] that the nearest neighboring control rule proposed in [15] solves the consensus problem for linear first-order integrator systems provided that their communication topology is characterized by an undirected and connected graph. This graph-based condition is relaxed in [13], [5] so that the network topology needs only to be a directed graph either with strong connectivity or of a spanning tree. More recently, a matrix-theoretical approach is developed in [10], [9] for the class of high-order linear dynamical systems (in particular, those of arbitrary but finite relative degrees), and it is shown that cooperative controllability is ensured if and only if the corresponding sensing/communication matrix sequence is sequentially complete.

On the other hand, most physical systems are nonlinear, and their nonlinear dynamics must be considered in both analysis and design. However, the combination of topological changes and nonlinear dynamics remains to be a major challenge although several kinds of results have been obtained so far. For example, if sensing/communication pattern of nonlinear systems is fixed, cooperative control can be addressed as a standard nonlinear control problem for which nonlinear techniques such as Lyapunov direct method can be successfully applied [4], [8]. Fixed pattern of information feedback or dynamics coupling among systems is assumed in qualitative analysis of the behavior of dynamical nonlinear circuit networks [1], in decentralized control [14], and in synchronization of coupled nonlinear systems with linear coupling [17], [16]. Recently, time-varying topological patterns become admissible in [7] for discrete systems whose dynamics satisfy a convexity property, and stability analysis is done using the combination of graph theory and discrete set-valued Lyapunov functions, which is also extended to continuous-time coupled nonlinear systems [6]. If communication is time varying but bidirectional, Lyapunov function can be found to design cooperative control for nonlinear systems [12].

In this paper, the consensus problem is further studied for nonlinear and heterogenous systems whose dynamics are not restricted to be convex and whose sensing and communications patterns are arbitrarily time varying. For for a linear cooperative system over any time finite interval, a quadratic cooperative control Lyapunov function can be found using averaging. It is shown that such a Lyapunov function always contains the same components which correspond to the consensus problem. For nonlinear systems with topology changes, a new stability result is established based on the properties of these Lyapunov function components and topology changes. This result does not require any convexity condition on system dynamics or involve any non-smooth analysis. Based on this result, a constructive design of nonlinear cooperative control is proposed and explicit stability conditions on nonlinear dynamics are obtained. Examples and their simulation are presented to illustrate the design process, to demonstrate its effectiveness, and to show performance of proposed nonlinear cooperative controls.

II. Problem Statement

Consider a collection of nonlinear heterogeneous dynamical systems whose dynamics are given by

$$\dot{y}_\mu = f_\mu(y_\mu) + g_\mu(y_\mu)u_\mu, \quad \mu = 1, \cdots, q, \tag{1}$$

where $y_\mu \in \Re^m$ is the state, matrix function $g_\mu(x_\mu)$ is assumed to be invertible (which is sufficient to ensure

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controllability and can be relaxed), and $u_\mu \in \Re^m$ is the cooperative control to be designed according to the available information from other dynamical systems.

A. Cooperative Systems and Their Consensus Problem

The control objective is to ensure that the systems in (1) are cooperative as a group and their outputs all reach a value of consensus, which is stated mathematically as follows. Systems of outputs y_μ (for $\mu = 1, \cdots, q$) are said to be *cooperatively stable* if, for every given $\epsilon > 0$, there exist some non-empty set Ω_0 and a constant $\delta > 0$ such that, for all the initial conditions satisfying $y_\mu(t_0) \in \Omega_0$ and $\|y_\mu(t_0) - y_l(t_0)\| \leq \delta$, $\|y_\mu(t) - y_l(t)\| \leq \epsilon$ for all $t \geq t_0$ and for all μ, l. The systems are said to be *asymptotically cooperatively stable* if they are cooperatively stable and if, for all μ and for all initial conditions $y_\mu(t_0) \in \Omega_0$, there exists a constant $c \in \Re$ such that $\lim_{t\to\infty} y_\mu(t) = c\mathbf{1}$, where $\mathbf{1}$ is the column vector of 1s. Three aspects are worth noting here. First, the limit c is a finite constant. In some applications, the control objective can be $\lim_{t\to\infty}[y_\mu(t) - y_l(t)] = 0$ while the limit of $y_\mu(t)$ may not be finite, for which the above definition could be relaxed to that $\lim_{t\to\infty}[y_\mu(t) - c(t)\mathbf{1}] = 0$ and the corresponding result could be called to be *weakly asymptotically cooperatively convergent*. Second, if all the dynamical systems are asymptotically stable, $\lim_{t\to\infty} y_\mu(t) = 0$, and the above definition is met with $c = 0$. This is the trivial case of cooperative control and the consensus problem, and the non-trivial and interesting cases are those for which $c \neq 0$ is not pre-determined. Third, asymptotical cooperative stability is global if $\Omega_0 = \Re^m$ and local if Ω_0 is a bounded set.

Different from standard control problems in which feedback is available for all the time (or at all sampling instants), cooperative systems often have intermittent information exchanges. The following sensing/communication matrix is to capture the information exchange among the systems:

$$S(t) = \begin{bmatrix} 1 & s_{12}(t) & \cdots & s_{1q}(t) \\ s_{21}(t) & 1 & \cdots & s_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{q1}(t) & s_{q2}(t) & \cdots & 1 \end{bmatrix} \tag{2}$$

where $s_{ij}(t) = 1$ if the information from the jth dynamical system is known to the ith system at time t, and $s_{ij}(t) = 0$ if otherwise. Without loss of any generality, it is assumed that changes of $S(t)$ occur at an infinite sequence of time instants, say, $\{t_k : \ k \in \aleph\}$ where $\aleph \triangleq \{0, 1, \cdots, \infty\}$. In other words, $S(t) = S(t_k)$ for all $t \in [t_k, t_{k+1})$. In order to account for dynamically changing environments, time sequence $\{t_k : \ k \in \aleph\}$ and corresponding changes in $S(t)$ are treated as uncertainties in cooperative control design and only used in stability analysis. The only knowledge available to the cooperative control design is that binary values of $s_{ij}(t)$ (for $j = 1, \cdots, q$) are known to the ith system immediately at time t but *not* apriori.

B. A Class of Cooperative Controls

The goal of this paper is to design a class of cooperative controls for the group of heterogeneous and nonlinear dynamical systems in (1) and their extensions (see section VI). Nonlinear cooperative control for the μth system is chosen to be of the general form:

$$\begin{aligned} u_\mu(t) &= g_\mu^{-1}(y_\mu(t)) \Big[-\alpha_\mu(y_\mu(t)) \\ &\quad + R_\mu(y_\mu, t) \sum_{i=1}^{q} D_{\mu i}(t) \beta_i(y_i(t), t) \Big], \end{aligned} \tag{3}$$

where $\alpha_\mu(y_\mu) : \Re^m \to \Re^m$ is the self feedback term, $R_\mu(y_\mu, t)$ is a gain matrix, $\beta_i(y_i, t) : \Re^m \times \Re^+ \to \Re^m$ is the local feedback term from the ith system, these functions are locally uniformly bounded with respect to argument y_μ and also uniformly bounded with respect to time t. To account for the availability of feedback information, the local feedback $\beta_i(y_i, t)$ in (3) is weighted by $D_{\mu i}(t)$, where

$$D_{\mu i}(t) = \frac{s_{\mu i}(t)}{\sum_{\nu=1}^{q} s_{\mu\nu}(t)} K_c, \tag{4}$$

$s_{\mu i}(t)$ is the (μ, i)-th element of sensing/communication matrix $S(t)$ defined in (2), the binary values of $s_{\mu i}(t)$ characterize whether the ith system is within the sensing/communication range of the μth system, and K_c is a constant, non-negative row stochastic and irreducible matrix. In what follows, functions $\alpha_\mu(\cdot)$, $R_\mu(\cdot)$, and $\beta_\mu(\cdot)$ will be synthesized to ensure cooperative stability provided that their sensing/communication makes them cooperatively controllable as a group. The proposed design is based on Lyapunov direct method, and the corresponding Lyapunov function components are sought by first studying linear cooperative systems.

III. Analysis and Design of Linear Cooperative Systems

To explain concepts and develop the tools and results necessary for subsequent analysis and design of heterogeneous systems, let us begin with the simplest systems whose dynamics are $f_\mu = 0$ and $g_\mu = 1$ in equation (1). That is, dynamical systems are all identical, linear, and affine as

$$\dot{y}_\mu = u_\mu. \tag{5}$$

In this case, cooperative control (3) would become

$$u_\mu(t) = -k_0 y_\mu(t) + \sum_{i=1}^{q} D_{\mu i}(t) y_i(t), \tag{6}$$

where $k_0 > 0$ is a constant gain to be chosen by the designer. Accordingly, the closed loop system is

$$\dot{y} = [-k_0 I + D(t)]y, \quad y(0) \text{ given}, \quad t \geq 0, \tag{7}$$

where

$$D(t) = \begin{bmatrix} D_{11}(t) & \cdots & D_{1q}(t) \\ \vdots & & \vdots \\ D_{q1}(t) & \cdots & D_{qq}(t) \end{bmatrix},$$

and its asymptotical stability and asymptotical cooperative stability are summarized in the subsequent subsections.

A. Asymptotic Stability

It is straightforward to study asymptotic stability of system (7) using Lyapunov function $V(y) = y^T y$ and the fact that $D(t)$ is nonnegative and row stochastic, and the result is summarized into the following lemma.

Lemma 1: Consider networked system (7). Then, it is globally asymptotically stable if $k_0 > 1$. If $k_0 < 1$, the system is unstable in the sense that its state will grow unbounded. If $k_0 = 1$ and if $D(t) = I$, the system is Lyapunov stable but not asymptotically stable.

As mentioned before, asymptotic stability under the choice of $k_0 > 1$ implies asymptotical cooperative stability. However, this result is trivial since the stability is achieved with or without any cooperation (i.e., the latter case is that $S(t) = I$) and since the equilibrium is uniquely prescribed. The choice of $k_0 = 1$ is a bifurcation value at which asymptotic cooperative stability becomes non-trivial and interesting. It is both intuitive and simple to show that, if a subgroup of the systems are insulated from the rest of systems, both subgroups of the systems are not asymptotically stable or cooperative asymptotically stable.

B. Asymptotical Cooperative Stability

Asymptotical cooperative stability of the overall linear networked system (7) with $k_0 = 1$ has been studied using graph-theoretical approach [3] and matrix-theoretical method [10]. The two techniques provide consistent and conceptually complementary stability conditions and, in comparison, the matrix-theoretical method has several advantages: it can handle heterogenous systems, it is well connected to other control methodologies including nonlinear techniques, and it can be applied to nonlinear systems as will be shown in this paper. Accordingly, the key concepts and main results in [10] are briefly described here.

The matrix-theoretical method fully exploits the properties of matrix $D(t)$ in (7): it is non-negative, row stochastic, piecewise constant; and its topological properties are identical to those of matrix $S(t)$ since $D_{\mu i}(t)s_{\mu i}(t) = D_{\mu i}(t)$ and since K_c is irreducible. The topological properties of matrices $S(t)$ and $D(t)$ are connectivity and grouping properties captured in general by the following lower-triangular canonical form [2], that is, there exists a piecewise constant permutation matrix $\mathcal{T}$ such that

$$\mathcal{T}^T D \mathcal{T} = \left[\begin{array}{cccc} E_{11} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{p1} & E_{p2} & \cdots & E_{pp} \end{array}\right] \stackrel{\triangle}{=} E_{\lrcorner}, \qquad (8)$$

where $1 \le p \le qm$, $E_{ii} \in \Re^{r_i \times r_i}$ is either a scalar or a square and irreducible sub-matrix of dimension higher than 1, and $qm = r_1 + \cdots + r_p$. Structure of matrix $E_{\lrcorner}$ reveals topological properties of system (7). In particular, matrix $D(t)$ at time t is said to be *lower triangularly complete* if, in (8) and for every $i \ge 2$, there exists at least one $j < i$ such that $E_{ij} \neq 0$. Matrix $D(t)$ or $S(t)$ being lower triangularly complete means that physically the systems corresponding to block E_{11} act as the instantaneous leaders for the rest systems and that graphically the corresponding directed graph has a *globally reachable node*.

Due to the fact that matrix $S(t)$ and consequently $D(t)$ in (7) vary over time and their changes cannot be modeled or prescribed in control design, cooperative stability condition should depend upon the cumulative effect of topology changes. To this end, let us introduce the following binary product of sensor/communication matrix sequence $S(t_k)$ over consecutive time intervals: for any given subsequence $\mathcal{I}' \stackrel{\triangle}{=} \{k'_v : v \in \aleph\}$ of $\aleph$, the cumulative exchange of information over time interval $[t_{k'_v}, t_{k'_{v+1}}]$ is described as

$$S_\Lambda(t_{k'_v}, t_{k'_{v+1}}) \stackrel{\triangle}{=} S(t_{k'_{v+1}}) \bigwedge S(t_{k'_{v+1}-1}) \bigwedge \cdots \bigwedge S(t_{k'_v}), \qquad (9)$$

where $\bigwedge$ denotes the operation of generating a binary product of two binary matrices. In parallel, the composite state transient matrix over the same interval is the matrix log function of the product of state transient matrices corresponding to matrices $[-I + D(t_{k'_{v+1}})]$ down to $[-I + D(t_{k'_v})]$.

It follows that the composite topology matrix $S_\Lambda(t_{k'_v}, t_{k'_{v+1}})$ over the composite time interval $[t_{k'_v}, t_{k'_{v+1}}]$ also has its lower-triangular canonical form in the form of (8). Hence, we can check whether matrix $S_\Lambda(t_{k'_v}, t_{k'_{v+1}})$ is lower triangularly complete. It is defined in [10] that sensor/communication matrix sequence $\{S(t_k), k \in \aleph\}$ given by (2) is said to be *sequentially complete* if there exists an infinitely-long subsequence $\mathcal{I}' \stackrel{\triangle}{=} \{k'_v : v \in \aleph\} \subset \aleph$ such that, over every time interval $[t_{k'_v}, t_{k'_{v+1}}]$, matrix $S_\Lambda(t_{k'_v}, t_{k'_{v+1}})$ is lower triangularly complete. As stated in the following theorem, the sequence of sensing/communication patterns being sequential complete is necessary and sufficient for cooperative controllability of heterogeneous linear systems (see [10] for the details about systems of different relative degrees, while systems in (5) are chosen to be homogeneous for simplicity).

Theorem 1: [10] Networked system (7) with $k_0 = 1$ is globally asymptotically cooperatively stable if and only if its sensor/communication matrix sequence $\{S(t_k), k \in \aleph\}$ is sequentially complete.

To extend this result to nonlinear systems, we need to develop first a corresponding Lyapunov argument on asymptotic cooperative stability of linear systems, which is the subject of the next subsection. This subsection ends with the following example which illustrates the concept of sequentially complete sensor/communication sequences.

Example 1: Let us consider the sensor/communication sequence $\{S(t_k), k \in \aleph\}$ defined by $S(t_k) = S_1$ for $k = 4\eta$, $S(t_k) = S_2$ for $k = 4\eta + 1$, $S(t_k) = S_3$ for $k = 4\eta + 2$, and $S(t_k) = S_4$ for $k = 4\eta + 3$, where $\eta \in \aleph$,

$$S_1 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad S_2 = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and } S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

It follows that

$$S_1 \bigwedge S_2 \bigwedge S_3 \bigwedge S_4 = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \triangleq \left[\begin{array}{c|c} S'_{\Lambda,11} & \emptyset \\ \hline S'_{\Lambda,21} & 1 \end{array}\right],$$

and its sequential completeness of sequence is obvious. $\triangle$

C. Cooperative Control Lyapunov Function

Analysis and control design can be done in terms of control Lyapunov function [4]. To this end, the so-called cooperative control Lyapunov function is defined below.

Definition 1: $V_c(x(t), t, t')$ is said to be a *cooperative control Lyapunov function* for a networked system $\dot{x} = F(x, t)$ with initial condition $x(t')$ if $V_c(x, t, t')$ is uniformly bounded with respect to t and t', if $V_c(x(t'), t', t') > V_c(x(t), t, t')$ holds along any solution of the system and for all $t > t'$, and if, for all $c \in \Re$, $F(c\mathbf{1}, t) = 0$, $V_c(c\mathbf{1}, t, t') = 0$ is the global minimum.

For cooperative control of linear systems, we need to search for a cooperative control Lyapunov function of form

$$V_c(y) = \sum_{\mu,k=1,\mu\neq k}^{q} p_{\mu k} \|y_\mu - y_k\|^2, \quad (11)$$

where $p_{\mu k} > 0$ are scalar constants. Function $V_c(y)$ in (11) has a global minimum corresponding to the solution to the consensus problem. The following theorem provides existence of $V_c(x(t), t, t_k)$ during a specific time interval over which $D(t) = D(k)$ is constant, and it provides an one-to-one relationship between topological properties exposed in (8) and existence of cooperative control Lyapunov function.

Theorem 2: [12], [11] Consider networked system (7) with $k_0 = 1$ and over time interval $t \in [t_k, t_{k+1})$ during which canonical form $E_{\lrcorner}$ in (8) can be obtained from constant matrix $D(t)$. Then, cooperative control Lyapunov function $V_c(y, t, t_k)$ of form (11) exists if and only if either $E_{\lrcorner}$ is irreducible (i.e., $p = 1$) or $E_{\lrcorner}$ is reducible (i.e., $p > 1$) and lower triangularly complete.

Theorem 2 provides the building block for us to search for Lyapunov function as topology changes over time. In particular, a composite Lyapunov function can now be sought over consecutive intervals over which cumulative effects of the topological changes meet the conditions in theorem 2. It is through this approach that the following equivalence is established among the sequential completeness property on time varying sensor/communication topologies, asymptotical cooperative stability, and existence of cooperative control Lyapunov function.

Theorem 3: [11] Consider the closed loop networked linear dynamical system (7) with $k_0 = 1$. Then, the following statements are equivalent:

(i) The sensor/communication sequence $\{S(t_k), k \in \aleph\}$ is sequentially complete (with respect to $\mathcal{I}' \triangleq \{k'_v : v \in \aleph\}$, an infinitely-long subsequence of $\aleph$).

(ii) System (7) is asymptotically cooperatively stable.

(iii) System (7) has a quadratic cooperative control Lyapunov function $V_c(y)$ of form (11) over every time interval $[t_{k'_v}, t_{k'_{v+1}}]$ or any their consecutive union.

In [11], theorem 3 is proven by determining an average system of system dynamics over time interval $[t_{k'_v}, t_{k'_{v+1}}]$ and by applying theorems 2 and 1. In other words, cooperative control Lyapunov function $V_c(y)$ is found in terms of the average system and based on the cumulative topological property. Since a nonlinear system with topology changes does not generally have an invariant average system over a composite time interval, this approach cannot be directly applied. Nonetheless, in search of cooperative control Lyapunov function, we must consider both dynamics of individual systems *and* cumulative topological property of the overall networked system. To overcome this fundamental difficulty, we propose to find not a scalar Lyapunov function but a collection of Lyapunov function components. In the next section, we will develop a cooperative stability result by studying the properties of Lyapunov function components $\|y_i - y_j\|^2$ (which are terms contained in the Lyapunov function $V_c(y)$ in (11)).

IV. COOPERATIVE STABILITY RESULT OF NONLINEAR SYSTEMS

It is well known that the search for Lyapunov function V_c is done by a backward procedure of first making $\dot{V}_c$ negative definite and then solving for V_c [4], [8]. For nonlinear systems, completing the procedure is often very difficult for a standard control problem. In an application to cooperative systems, the procedure becomes even more difficult because changes of sensing/communication cannot be assumed in stability analysis and control design and, as shown in the last section, the cumulative property of their changes over time determines cooperative stability. To overcome this difficulty, we investigate cooperative stability by analyzing the properties of Lyapunov function components. This new approach allows us to avoid the search of cooperative Lyapunov function as the Lyapunov function components can provide qualitative properties sufficient for concluding cooperative stability. The following theorem represents the first result along this direction and, because of space limitation, its extensions are beyond the scope of this paper and its proof is limited to the case that $S(t)$ (and hence $D(t)$) is lower triangularly complete.

Theorem 4: Consider a general system of form $\dot{\xi} = \mathcal{F}(\xi, D(t))$, where $D(t)$ is a non-negative, piecewise-constant and row-stochastic matrix, and $\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_n \end{bmatrix}^T \in \Re^n$. The system is asymptotically cooperatively stable if the following conditions hold:

(i) Along any trajectory of the system, the time derivatives of Lyapunov function components $|\xi_\mu - \xi_k|^2$ satisfy the following inequalities: for $\mu, k \in \Omega$,

$$\begin{aligned} \frac{d|\xi_\mu - \xi_k|^2}{dt} \leq & -2|\xi_\mu - \xi_k|^2 + 2\sum_{l=1}^{n} (\xi_\mu - \xi_k) \\ & \times [d_{\mu l}(t) - d_{kl}(t)]\xi_l(t), \end{aligned} \quad (12)$$

where $d_{\mu k}(t)$ are the scalar elements of matrix $D(t)$, $d_{ij}(t)$ is uniformly bounded away from zero whenever $d_{ij}(t) \neq 0$, and $\Omega = \{1, \cdots, n\}$ is the set of indices.

(ii) The sequence $\{S(t_k), k \in \aleph\}$ corresponding to $D(t)$ is sequentially complete.

Proof for the case that $D(t)$ is time varying but lower triangularly complete: Define the following subsets of Ω: at any time t,

$$\Omega_{max}(t) = \{i \in \Omega \,:\, \xi_i = \xi_{max}\}\,, \quad \xi_{max}(t) = \max_{j\in\Omega} \xi_j(t),$$

$$\Omega_{min}(t) = \{i \in \Omega \,:\, \xi_i = \xi_{min}\}\,, \quad \xi_{min}(t) = \min_{j\in\Omega} \xi_j(t),$$

$$\Omega_{mid}(t) = \{i \in \Omega \,:\, \xi_{min} < \xi_i < \xi_{max}\}\,.$$

It is apparent that, unless $\xi_i = \xi_j$ for all i and j, $\xi_{min} < \xi_{max}$ and set Ω is partitioned into the three mutually disjoint subsets of Ω_{max}, Ω_{mid} and Ω_{min}. Also, let us define the maximum relative distance as $\delta_{max}(t) = \max_{\mu,k\in\Omega} |\xi_\mu(t) - \xi_k(t)|$. It is obvious that $\delta_{max}(t) = \xi_{max}(t) - \xi_{min}(t)$.

It follows from (12) and from $D(t)$ being row stochastic that

$$\begin{aligned}
&\frac{d}{dt}|\xi_\mu - \xi_k|^2 \\
\leq&-2|\xi_\mu - \xi_k|^2 + 2\sum_{l=1}^{n}(\xi_\mu - \xi_k)[d_{\mu l}(t) - d_{kl}(t)](\xi_l - \xi_k) \\
=&-2|\xi_\mu - \xi_k|^2 + 2(\xi_\mu - \xi_k)\sum_{l=1}^{n} d_{\mu l}(t)(\xi_l - \xi_k) \\
&-2(\xi_\mu - \xi_k)\sum_{l=1}^{n} d_{kl}(t)(\xi_l - \xi_k). \qquad (13)
\end{aligned}$$

Thus, for any $\mu^* \in \Omega_{max}$, $k^* \in \Omega_{min}$, and for all $l \in \Omega$,

$$\xi_{\mu^*} - \xi_{k^*} \geq 0, \quad \xi_l - \xi_{k^*} \geq 0.$$

Since matrix $D(t)$ is non-negative and row stochastic,

$$-2(\xi_{\mu^*} - \xi_{k^*})\sum_{l=1}^{n} d_{k^* l}(t)(\xi_l - \xi_{k^*}) \leq 0, \qquad (14)$$

and

$$0 \leq (\xi_{\mu^*} - \xi_{k^*})\sum_{l=1}^{n} d_{\mu^* l}(t)(\xi_l - \xi_{k^*}) \leq |\xi_{\mu^*} - \xi_{k^*}|^2. \quad (15)$$

Substituting the above inequalities into (13) yields $d|\xi_{\mu^*} - \xi_{k^*}|^2/dt \leq 0$. To show asymptotic cooperative stability, it is sufficient to show that $d|\xi_{\mu^*} - \xi_{k^*}|^2/dt < 0$. To this end, we need only show that at least one of inequalities (14) and (15) is a strict inequality. To prove our proposition by contradiction, let us assume that both (14) and (15) be equalities. Then, unless $\xi_{min} = \xi_{max}$, the equality of (14) implies $d_{k^* l}(t) = 0$ for $l \in \Omega_{mid} \cup \Omega_{max}$ and $k^* \in \Omega_{min}$, and (15) being an equality means $d_{\mu^* l}(t) = 0$ for $l \in \Omega_{mid} \cup \Omega_{min}$ and $\mu^* \in \Omega_{max}$. This means that, unless $\xi_{min} = \xi_{max}$, there is permutation matrix $P(t)$ under which

$$P(t)D(t)P^T(t) = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \triangleq E(t), \quad (16)$$

where E_{ii} are square blocks, row indices of E_{11} correspond to those in Ω_{min}, row indices of E_{22} correspond to those in Ω_{max}, and row indices of E_{33} correspond to those in Ω_{mid}. Clearly, the structure of matrix $E(t)$ contradicts with the knowledge that $D(t)$ is lower triangularly complete. Hence, we know that either (14) or (15) or both must be a strict inequality. □

The results in theorem 4 have several distinct and important features. First, cooperative stability is concluded without the exact knowledge of the corresponding Lyapunov function. Instead, qualitative properties of Lyapunov function components are used. Second, the Lyapunov argument directly utilizes the topological property of the networked system. Third, despite of instantaneous changes by the sensing/communication network, the Lyapunov argument does not involve any non-smooth analysis. Inequality (12) can easily be verified using dynamic equations of the system. Fourth, there is no need to make such assumption as convexity about the solution of nonlinear systems, and cooperative stability is concluded for nonlinear systems in general. Finally, the above theorem includes as a special case cooperative control of linear systems and the existing results summarized in the last section.

V. DESIGN AND ANALYSIS OF NONLINEAR COOPERATIVE CONTROLS

In this section, design of nonlinear cooperative control are pursued for nonlinear systems (1) by applying theorem 4. To this end, feedback functions $\alpha_\mu(\cdot)$, $R_\mu(\cdot)$, and $\beta_i(\cdot)$ in (3) are the critical choices to achieve cooperative stability and to enhance convergence of the consensus problem. Functions $\beta_\mu(\cdot)$ should be chosen to be differentiable and have the property that, for all $\mu \in \{1, \cdots, q\}$ and for all t,

$$R_\mu(y_\mu, t) = \left[\frac{\partial \beta_\mu(y_\mu, t)}{\partial y_\mu}\right]^{-1} \text{ exists for all } y_\mu, \qquad (17)$$

and

$$\lim_{t\to\infty} \beta_\mu(y_\mu(t), t) = c_1\mathbf{1} \quad \text{implies} \quad \lim_{t\to\infty} y_\mu(t) = c_2\mathbf{1}, \;(18)$$

where c_1, c_2 are arbitrary scalar constants. Upon choosing $\beta_\mu(\cdot)$, functions $\alpha_\mu(\cdot)$ should be chosen such that inequality

$$\begin{aligned}
&[\beta_\mu(y_\mu, t) - \beta_k(y_k, t)]^T \left\{ \frac{\partial \beta_\mu(y_\mu, t)}{\partial y_\mu}[f_\mu(y_\mu) - \alpha_\mu(y_\mu)] \right. \\
&-\frac{\partial \beta_k(y_k, t)}{\partial y_k}[f_k(y_k) - \alpha_k(y_k)] + \frac{\partial \beta_\mu(y_\mu, t)}{\partial t} \\
&\left. -\frac{\partial \beta_k(y_k, t)}{\partial t} \right\} \\
\leq& -\|\beta_\mu(y_\mu, t) - \beta_k(y_k, t)\|^2 \qquad (19)
\end{aligned}$$

holds for all μ, k and t. Cooperative stability under nonlinear cooperative control (3) is ensured by the following theorem.

Theorem 5: Consider the heterogeneous and nonlinear dynamical systems in (1) and under nonlinear cooperative control (3). Then, if properties (17), (18), and (19) are satisfied through the choices of $\alpha_\mu(\cdot)$, $R_\mu(\cdot)$, and $\beta_i(\cdot)$, the collection of dynamical systems as a group are asymptotically cooperatively stable provided that the sensor/commnunication sequence $\{S(t_k), k \in \aleph\}$ is sequentially complete.

Proof: Under control (3), the systems in (1) become

$$\dot{y}_\mu = f_\mu(y_\mu) - \alpha_\mu(y_\mu(t)) + R_\mu(y_\mu, t) \sum_{i=1}^{q} D_{\mu i}(t) \beta_i(y_i(t), t).$$

Let us define the new variables:

$$z_\mu(t) \stackrel{\triangle}{=} \beta_\mu(y_\mu(t), t), \tag{20}$$

where $\mu = 1, \cdots, q$. It follows that, for any pair (μ, k),

$$\begin{aligned} & \frac{d}{dt}\|z_\mu - z_k\|^2 \\ = \; & 2(z_\mu - z_k)^T \left\{ \frac{\partial \beta_\mu(y_\mu, t)}{\partial y_\mu} [f_\mu(y_\mu) - \alpha_\mu(y_\mu)] \right. \\ & - \frac{\partial \beta_k(y_k, t)}{\partial y_k} [f_k(y_k) - \alpha_k(y_k)] + \frac{\partial \beta_\mu(y_\mu, t)}{\partial t} \\ & \left. - \frac{\partial \beta_k(y_k, t)}{\partial t} \right\} + 2 \sum_{i=1}^{q} (z_\mu - z_k)^T [D_{\mu i}(t) \\ & - D_{ki}(t)] z_i(t). \end{aligned}$$

Hence, it follows from inequality (19) that

$$\begin{aligned} \frac{d}{dt}\|z_\mu - z_k\|^2 \le -2\|z_\mu - z_k\|^2 + 2 \sum_{i=1}^{q} (z_\mu - z_k)^T \\ \times [D_{\mu i}(t) - D_{ki}(t)] z_i(t). \end{aligned} \tag{21}$$

Now, consider the resulting system $\dot{z} = \mathcal{F}(z, S(t))$ obtained under the state transformation (20). It follows from theorem 4 and from (21) that $z_\mu = \beta_\mu(y_\mu) \to c_1 \mathbf{1}$ for some scalar constant c_1. Finally, cooperative stability of $y_\mu \to c_2 \mathbf{1}$ to the consensus problem can be concluded for some scalar constant c_2 by invoking property (18). □

Two observations are worth mentioning about properties (17) and (18). First, property (17) ensures that function $\beta(y_\mu(t), t)$ has an inverse, and property (18) states that there is a special pair of solution in the limit. If $\beta(y_\mu(t), t)$ is chosen to be locally invertible (such as periodic function $\beta(y_\mu(t), t) = \sin(y_\mu(t))$), $R_\mu(\cdot)$ is locally defined, and theorem 5 still holds in the sense that $\beta(y_\mu(t), t)$ is globally asymptotically cooperatively stable but $y_\mu(t)$ may only be locally asymptotically cooperatively stable. Second, property (18) implies that, as $t \to \infty$, $\beta_\mu(y_\mu(t), t)$ approaches some function $\beta(y_\mu(t))$ for all $\mu \in \{1, \cdots, q\}$, at least locally. In other words, the simplest choice of the local feedback terms in the cooperative control would be identical. If $\beta_\mu(\cdot)$ are independent of time, they have to be the same locally. Nonetheless, it may be of advantage that, during the transient, feedback information from various sensors and communication channels are weighted differently in order to improve convergence and robustness.

In the presence of additive noises and disturbances, the overall closed loop system of systems (1) under control (3) is

$$\dot{y} = [f(y) - A(y)] + R(y, t) D(t) B(y, t) + w(t),$$

where

$$\left\{ \begin{array}{l} f(y) = [f_1^T(y_1) \; \cdots \; f_q^T(y_q)]^T \\ A(y) = [\alpha_1^T(y_1) \; \cdots \; \alpha_q^T(y_q)]^T \\ R(y, t) = \mathrm{diag}\{R_1(y_1, t), \cdots, R_q(y_q, t)\} \\ B(y, t) = [\beta_1^T(y_1, t) \; \cdots \; \beta_q^T(y_q, t)]^T \\ w(t) = [w_1^T(t) \; \cdots \; w_q^T(t)]^T. \end{array} \right. .$$

It follows from standard control Lyapunov function $V = \|y\|^2$ and from input-to-state stability condition [4] that the system is uniformly bounded for all uniformly bounded noises $w_i(t)$ if the following condition holds: for all μ,

$$y_\mu^T [f_\mu(y_\mu) - \alpha_\mu(y_\mu)] \le -\gamma_\mu(\|y_\mu\|), \tag{22}$$

where $\gamma_\mu(\|y_\mu\|)$ is a positive definite function of higher order in $\|y_\mu\|$ than both $\|y_\mu\|^2$ and $\|R_\mu(y_\mu, t)\|^2 \cdot \|\beta_\mu(y_\mu, t)\|^2$. Condition (22) is a sufficient condition for robustness.

VI. Extension to Systems of Higher Relative Degrees

Nonlinear systems in (1) are of relative degree one. The proposed design and stability analysis can be extended to systems of higher relative degrees. To see what is involved in such an extension, consider the case that the collection of systems are

$$\left\{ \begin{array}{l} \dot{y}_\mu = f_\mu(y_\mu) + g_\mu(y_\mu) u_\mu, \quad \mu = 1, \cdots, (q-1), \\ \dot{y}_q = F_{q,1}(y_q, y_q') + G_{q,1}(y_q, y_q') H_q(y_q') \\ \dot{y}_q' = F_{q,2}(y_q, y_q') + G_{q,2}(y_q, y_q') u_q \end{array} \right. , \tag{23}$$

where $y_q, y_q' \in \Re^m$, $x_q = \begin{bmatrix} y_q^T & y_q' \end{bmatrix}^T$, and y_q' is not available to any system other than the qth system.

Due to the higher relative degree of the qth system, dynamics of the overall system are augmented. Accordingly, let us introduce the augmented matrices $\overline{S}(t)$ and $\overline{D}(t)$ as

$$\overline{S}(t) = \begin{bmatrix} S_{11}(t) & S_{12} & \cdots & S_{1q}(t) & 0 \\ \vdots & & & & \vdots \\ S_{(q-1)1}(t) & S_{(q-1)2} & \cdots & S_{(q-1)q}(t) & 0 \\ 0 & 0 & 0 & 0 & 1 \\ S_{q1}(t) & S_{q2}(t) & \cdots & S_{qq}(t) & 0 \end{bmatrix},$$

and

$$\overline{D}(t) = \begin{bmatrix} D_{11}(t) & D_{12} & \cdots & D_{1q}(t) & 0 \\ \vdots & & & & \vdots \\ D_{(q-1)1}(t) & D_{(q-1)2} & \cdots & D_{(q-1)q}(t) & 0 \\ 0 & 0 & 0 & 0 & 1 \\ D_{q1}(t) & D_{q2}(t) & \cdots & D_{qq}(t) & 0 \end{bmatrix}.$$

It has been shown in [10] that topological properties are invariant under the above augmentation from $S(t)$ to $\overline{S}(t)$.

Now, consider the collection of fictitious systems:

$$\left\{ \begin{array}{l} \dot{y}_\mu = f_\mu(y_\mu) + g_\mu(y_\mu) v_\mu, \quad \mu = 1, \cdots, (q-1), \\ \dot{y}_q = f_q(y_q, y_{q+1}) + g_q(y_q, y_{q+1}) v_q \\ \dot{y}_{q+1} = f_{q+1}(y_q, y_{q+1}) + g_{q+1}(y_q, y_{q+1}) v_{q+1} \end{array} \right. , \tag{24}$$

and fictitious cooperative controls

$$v_\mu(t) = g_\mu^{-1} \times \left[-\alpha_\mu' + R_\mu' \sum_{i=1}^{q+1} \overline{D}_{\mu i}(t) \beta_i(y_i(t), t) \right], \tag{25}$$

where

$$\begin{cases} f_q(y_q, y_{q+1}) = 0 \\ g_q(y_q, y_{q+1}) = 1 \\ f_{q+1}(y_q, y_{q+1}) = F_{q,2}(y_q, y_{q+1}) \\ g_{q+1}(y_q, y_{q+1}) = G_{q,2}(y_q, y_{q+1}) \end{cases}$$

and

$$\begin{cases} \alpha'_\mu = \alpha_\mu(y_\mu), \quad \mu = 1, \cdots, q-1 \\ R'_\mu = R_\mu(y_\mu, t), \quad \mu = 1, \cdots, q-1 \\ \alpha'_{q+1} = \alpha_q(y_q, y'_q) \\ R'_{q+1} = R_q(y_q, y'_q, t). \end{cases}$$

It follows from the entries in $\overline{D}(t)$ that

$$v_q(t) = -\alpha_q + R_q \beta_{q+1}(y_{q+1}(t), t)$$

and that controls $v_\mu(t)$ with $\mu \neq q$ are independent of y_{q+1}. Hence, systems (24) under cooperative controls (25) are identical to systems (23) under cooperative controls (3) provided that choices of $\alpha_q(\cdot)$ and $R_q(\cdot)$ are made according to the following equality:

$$\begin{aligned} & -\alpha_q(y_q, y'_q) + R_q(y_q, t)\beta_{q+1}(y'_q, t) \\ = \quad & F_{q,1}(y_q, y'_q) + G_{q,1}(y_q, y'_q) H_q(y'_q). \end{aligned} \tag{26}$$

Thus, cooperative control can be designed for systems (23) by applying theorem 5 to relative-degree-one systems (24).

VII. Examples and Simulation Results

In this section, four examples are presented to illustrate the proposed cooperative control design and performance of the nonlinear controllers designed. For simplicity, three systems (i.e., $q = 3$) are considered in each of the examples, and most of those systems are chosen to be scalar ($m = 1$).

It can be shown as did in example 1 that every sequence generated from four patterns in (10) becomes sequentially complete as long as the sequence contains infinite entries of all the patterns of S_1, S_2, S_3 and S_4. Thus, in the simulations of all the three examples, a sequentially complete sequence is generated randomly from S_1 up to S_4.

Example 2: Consider the collection of three systems whose dynamics are:

$$\dot{y}_\mu = -y_\mu^{\frac{1}{3}} + u_\mu, \quad \mu = 1, 2, 3, \tag{27}$$

where $y_\mu \in \Re$ and $u_\mu \in \Re$. It is easy to verify that, under the choices of

$$\alpha_\mu = y_\mu, \;\; \beta_\mu = y_\mu + y_\mu^3, \;\; \mu = 1, 2, 3,$$

properties (18) and (19) are globally satisfied. Hence, cooperative controls are

$$u_\mu = -y_\mu + \frac{1}{1 + 3y_\mu^2} \sum_{j=1}^{3} D_{\mu j}(t)(y_j + y_j^3). \tag{28}$$

Simulation of the overall closed loop nonlinear system is done with initial condition $[-1,\ 2,\ -2]^T$, and performance of systems (27) under cooperative control (28) is shown in figure 1. $\triangle$

Example 3: Consider the systems

$$\dot{\theta}_\mu = u_\mu,$$

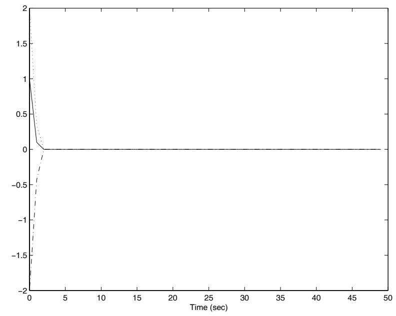

Fig. 1. Example 2: y_1 — solid line, y_2 — dotted line, y_3 — dashdot line

and cooperative controls

$$u_\mu = -\tan(\theta_\mu) + \frac{1}{\cos(\theta_\mu)} \sum_{j=1}^{3} D_{\mu j}(t) \sin(\theta_j), \tag{29}$$

where $\mu = 1, 2, 3$. The closed loop system can be viewed as a nonlinear version of Viscek's model [15].

It is straightforward to verify that, under the choices of $\alpha_\mu = \tan(\theta_\mu)$ and $\beta_\mu = \sin(\theta_\mu)$, property (18) is locally defined and properties (18) and (19) hold locally for $\theta_\mu \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, the overall system is locally asymptotically cooperatively stable. Figure 2 shows performance of cooperative control (29) for the initial condition $[\frac{\pi}{6},\ \frac{\pi}{4},\ -\frac{\pi}{4}]^T$.

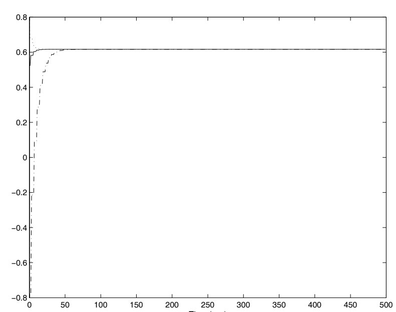

Fig. 2. Example 3: θ_1 — solid line, θ_2 — dotted line, θ_3 — dashdot line

In addition, it can be shown that, if initial conditions are not all in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and if there is no singularity, the original states y_μ may converge to different values while the transformed states z_μ are always asymptotically cooperatively stable. $\triangle$

Example 4: Consider the following heterogeneous systems:

$$\begin{cases} \dot{y}_1 = -\frac{y_1^3}{1+3y_1^2} + u_1 \\ \dot{y}_2 = -\tan(y_2) + u_2 \\ \dot{y}_3 = \frac{(e^{-y_3^2}-1)y_3}{2-e^{-y_3^2}+2y_3^2 e^{-y_3^2}} + (1+y_3^2)u_3 \end{cases}.$$

For illustration, cooperative control (3) can be implemented with the following choices:

$$\beta_1 = y_1 + y_1^3, \qquad \beta_2 = \sin(y_2), \quad \beta_3 = 2y_3 - y_3 e^{-y_3^2},$$

and

$$\alpha_1 = \frac{y_1}{1 + 3y_1^2}, \quad \alpha_2 = 0, \quad \alpha_3 = \frac{y_3}{2 - e^{-y_3^2} + 2y_3^2 e^{-y_3^2}}.$$

Since property (19) hold locally for $y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, transformed states $\beta_\mu(y_\mu)$ are asymptotically cooperatively stable. Figure 3 shows consensus of states y_μ, and figure 4 shows consensus of states β_μ, both of which are for initial condition $[\frac{\pi}{6},\ -\frac{\pi}{4},\ \frac{\pi}{4}]^T$. On the other hand, since functions $\beta_\mu(\cdot)$ are different, states y_μ do not reach consensus for all initial conditions (e.g., initial condition $[-\frac{\pi}{4},\ \frac{\pi}{4},\ \frac{\pi}{4}]^T$). $\triangle$

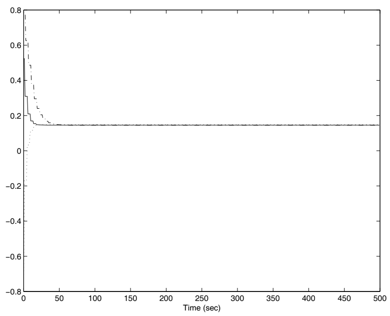

Fig. 3. Example 4: y_1 — solid line, y_2 — dotted line, y_3 — dashdot line

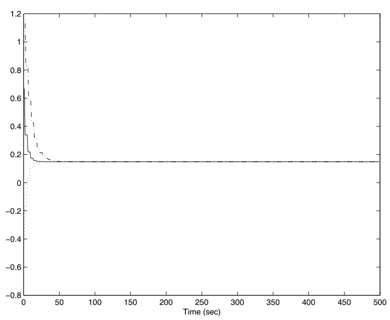

Fig. 4. Example 4: β_1 — solid line, β_2 — dotted line, β_3 — dashdot line

Example 5: Consider two relative-degree-one systems and one relative-degree-two system:

$$\begin{cases} \dot{y}_i = -y_i^3 + u_i, \quad i = 1, 2 \\ \dot{y}_3 = -y_3 - y_3^3 + \frac{y_3' + (y_3')^3}{1+3y_3^2}, \quad \dot{y}_3' = -(y_3')^3 + u_3 \end{cases}$$

Under the choice of $\beta_i = y_i + y_i^3$, cooperative controls (3) become

$$\begin{cases} u_i = -y_i + \frac{1}{1+3y_i^2} \sum_{j=1}^{3} D_{ij}(t)(y_j + y_j^3), \quad i = 1, 2, \\ u_3 = -y_3' + \frac{1}{1+(3y_3')^2} \sum_{j=1}^{3} D_{3j}(t)(y_j + y_j^3) \end{cases},$$

and all the properties are met. Simulation result under initial condition is $[-0.1, 0.2, -0.2, 0]^T$ is shown in figure 5. $\triangle$

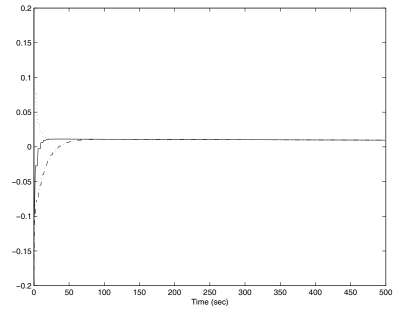

Fig. 5. Example 5: y_1 — solid line, y_2 — dotted line, y_3 — dashdot line

VIII. Conclusion

In this paper, the consensus problem of heterogeneous and nonlinear systems is studied. Cooperative stability of the systems is analyzed by Lyapunov direct method. Stability conditions are in terms of properties of Lyapunov function components and the sensing/communication network. The proposed Lyapunov-based design can be applied to arbitrary network topology and to synthesizing either feedback linearizing or nonlinear cooperative controls. Simulation results demonstrate effectiveness of the proposed design of nonlinear cooperative controls.

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