Project: Control of Frictional Dynamics Through Surface Vibration

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1 Control Frictional Dynamics of A One-Dimensional Particle Array

This section presents a worked out problem that the proposed project is based upon.

1.1 Problem Formulation

The following equation describes the dynamics of a particle array sliding on a surface subject to a periodical potential:

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = F_i + u(t) \tag{1}$$

where ϕ_i is the dimensionless phase variable, F_i is the nearest-neighbor interaction in the form of Morse-type interaction:

$$\begin{aligned} F_i = &\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right\} \\ &-\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right\}, \\ &i = 2, \ldots, N-1, \end{aligned} \tag{2}$$

κ and β are positive constants. The free-end boundary conditions are represented as:

$$\begin{aligned} F_1 &= \frac{\kappa}{\beta}\left\{e^{-\beta(\phi_2-\phi_1)} - e^{-2\beta(\phi_2-\phi_1)}\right\}, \\ F_N &= -\frac{\kappa}{\beta}\left\{e^{-\beta(\phi_N-\phi_{N-1})} - e^{-2\beta(\phi_N-\phi_{N-1})}\right\}. \end{aligned} \tag{3}$$

As $\beta \to 0$, (2) turns to:

$$F_i = \kappa\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right), \quad i = 2, \ldots, N-1, \tag{4}$$

which represents a linear approximation of particle interaction for small β with the following free-end boundary conditions:

$$F_1 = \kappa(\phi_2 - \phi_1), \; F_N = \kappa(\phi_{N-1} - \phi_N). \tag{5}$$

Due to physical accessibility constraints, the feedback control $u(t)$ is a function of three measurable quantities, $v_{target}, v_{c.m.}$, and $\phi_{c.m.}$, where v_{target} is the constant targeted velocity for the center of mass, $v_{c.m.}$ is the average (center of mass) velocity, *i.e.*,

$$v_{c.m.} = \frac{1}{N}\sum_{i=1}^{N}\dot{\phi}_i, \tag{6}$$

and $\phi_{c.m.}$ is the average (center of mass) position, *i.e.*,

$$\phi_{c.m.} = \frac{1}{N}\sum_{i=1}^{N}\phi_i. \tag{7}$$

We define the following tracking control problem:

Design a feasible feedback control law

$$u(t) = u(v_{target}, v_{c.m.}, \phi_{c.m.}), \tag{8}$$

such that $v_{c.m.}$ tends to v_{target}.

1.2 Tracking Control of the Average System

In this subsection, we design tracking control to solve the problem defined above, which is to render the average velocity of the system, *i.e.* the velocity of the center of the mass, to converge to a constant targeted value. To this end, we introduce the average error states as:

$$e_{1av} = \phi_{c.m.} - v_{target}t, \; e_{2av} = v_{c.m.} - v_{target}, \tag{9}$$

where $v_{c.m.}$ and $\phi_{c.m.}$ are defined in (6) and (7), respectively. Then it is obvious that the convergence of

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$(\phi_{c.m.}, v_{c.m.})$ to $(v_{target}t, v_{target})$ is equivalent to the convergence of (e_{1av}, e_{2av}) to $(0, 0)$. Therefore, asymptotic stability of the system in the error state space is equivalent to asymptotic tracking of the targeted positions and constant velocity.

The dynamics of (e_{1av}, e_{2av}) can be derived as:

$$\begin{aligned}\dot{e}_{1av} &= e_{2av}\\ \dot{e}_{2av} &= -\frac{1}{N}\sum_{i=1}^{N}\sin(e_{i1}+v_{target}t) - \gamma(e_{2av}+v_{target})\\ &\quad +u(t)\end{aligned} \tag{10}$$

Note that the F_i term disappeared in (10) because the sum of F_i is zero for Morse-type interactions of the form defined in (2).

We construct the following Lyapunov function candidate:

$$W(e_{av}) = \frac{1}{2}e_{1av}^2 + \frac{1}{2}(c_1 e_{1av} + e_{2av})^2 \tag{11}$$

where c_1 is a positive design constant, and $e_{av} = [e_{1av} \;\; e_{2av}]^T$.

Taking the time derivative of W along the dynamics of (10), and denoting

$$\xi = c_1 e_{1av} + e_{2av}, \tag{12}$$

we have:

$$\begin{aligned}\dot{W}(e_{av}) &= -c_1 e_{1av}^2 + \xi\,[e_{1av} + c_1 e_{2av} - \gamma e_{2av}\\ &\quad -\frac{1}{N}\sum_{i=1}^{N}\sin(e_{i1}+v_{target}t) - \gamma v_{target} + u(t)]\,.\end{aligned} \tag{13}$$

Choose

$$\begin{aligned}u(t) &= \gamma v_{target} - e_{1av} - (c_1-\gamma)e_{2av}\\ &\quad -(c_1+c_2)\xi + \sin(v_{target}t)\\ &= \gamma v_{target} - k_1(\phi_{c.m.} - v_{target}t)\\ &\quad -k_2(v_{c.m.} - v_{target}) + \sin(v_{target}t)\end{aligned} \tag{14}$$

where c_2 is a positive design constant, $k_1 = 1 + (c_1 + c_2)c_1, k_2 = 2c_1 + c_2 - \gamma$, and the term $\sin(v_{target}t)$ is introduced to enforce the equilibrium of the closed-loop system (10) to be the origin.

We obtain:

$$\begin{aligned}\dot{W}(e_{av}) &= -c_1(e_{1av}^2+\xi^2) - c_2\xi^2\\ &\quad +\xi\frac{1}{N}\sum_{i=1}^{N}[-\sin(e_{i1}+v_{target}t) + \sin(v_{target}t)]\\ &\le -c_1(e_{1av}^2+\xi^2) - c_2\xi^2 + |\xi|\\ &\quad \cdot\frac{1}{N}\sum_{i=1}^{N}|-\sin(e_{i1}+v_{target}t) + \sin(v_{target}t)|\\ &\le -c_1(e_{1av}^2+\xi^2) - c_2\xi^2 + 2|\xi|\end{aligned} \tag{15}$$

Since the maximum of the last two terms is $1/c_2$, we have

$$\dot{W}(e_{av}) \le -c_1(e_{1av}^2+\xi^2) + \frac{1}{c_2}, \tag{16}$$

which can be used to prove uniform boundedness of the error system (10) as shown in the proof of Theorem 1.

To achieve asymptotical tracking, that is, to make the error system (10) asymptotically stable, the following switching control law can be used:

$$\begin{aligned}u(t) &= \gamma v_{target} - k_1(\phi_{c.m.} - v_{target}t) - k_2\,(v_{c.m.} - v_{target})\\ &\quad + \sin(v_{target}t) - 2sgn(\xi)\end{aligned} \tag{17}$$

where $sgn(\xi)$ denotes the signum function, defined as $sgn(\xi) = 1$ for $\xi > 0$, $sgn(\xi) = -1$ for $\xi < 0$, and $sgn(\xi) = 0$ for $\xi = 0$.

The following theorem presents the stability results of the closed-loop average error system (10).

Theorem 1 *The feedback control laws (14) or (17) solve the tracking control of the average system defined in Subsection 1.1. Using (14), the tracking error between the velocity of the center of mass and the targeted velocity is uniformly bounded over time* $[0,\infty)$*. Under the switching control law (17), the tracking error goes to zero asymptotically.*

Proof: Using the continuous control law (14), for the positive definite Laypunov function W defined in (11), we obtained (16). Then,

$$\dot{W}(e_{av}) \le 0,\ \forall\|(e_{1av},\xi)\| \ge \frac{1}{\sqrt{c_1 c_2}}. \tag{18}$$

We conclude that the solutions of the closed-loop system (10), (14) are globally uniformly bounded.

To calculate the ultimate bound, we notice from (11) that

$$\begin{aligned}\frac{1}{2}\lambda_{min}(P)\|e_{av}\|^2 \le W(e_{av}) &= \frac{1}{2}e_{av}^T P e_{av}\\ &\le \frac{1}{2}\lambda_{max}(P)\|e_{av}\|^2\end{aligned} \tag{19}$$

where $e_{av} = [e_{1av}\ e_{2av}]^T$,

$$P = \begin{bmatrix} 1+c_1^2 & c_1\\ c_1 & 1\end{bmatrix},$$

and $\lambda_{min}(P)$, and $\lambda_{max}(P)$ denote the minimum and maximum eigenvalues of the matrix P, respectively. From (19), we have

$$\|e_{av}\|^2 \leq \frac{2W(e_{av})}{\lambda_{min}(P)} = \frac{\|(e_{1av}, \xi)\|^2}{\lambda_{min}(P)}. \tag{20}$$

Due to (18), we obtain

$$\dot{W}(e_{av}) \leq 0, \ \forall \|e_{av}\| \geq \frac{1}{\sqrt{c_1 c_2 \lambda_{min}(P)}}. \tag{21}$$

The ultimate bound of $\|e_{av}\|$ is given by:

$$b = \sqrt{\frac{\lambda_{max}(P)}{c_1 c_2 \lambda_{min}^2(P)}}. \tag{22}$$

By choosing c_1, c_2 appropriately (with the price of a large control effort), we can have the error states to be arbitrarily close to zero.

Under the switching control law (17) (which is the continuous control (14) plus a switching term), substituting (17) into (13), we get

$$\begin{aligned}\dot{W}(e_{av}) &\leq -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 + 2|\xi| - \xi 2sgn(\xi)\\ &\leq -c_1(e_{1av}^2 + \xi^2),\end{aligned} \tag{23}$$

which is negative definite. Asymptotic stability of the error system follows from Lyapunov theory. □

1.3 Simulation Results

Figures 1 and 2 demonstrate the tracking performances of the average system using the control law (14) with different initial conditions and for two different targeted values $v_{target} = 3, 1.5$ respectively. The system parameters are chosen to be

$$N = 3, \kappa = 0.26, \gamma = 0.1.$$

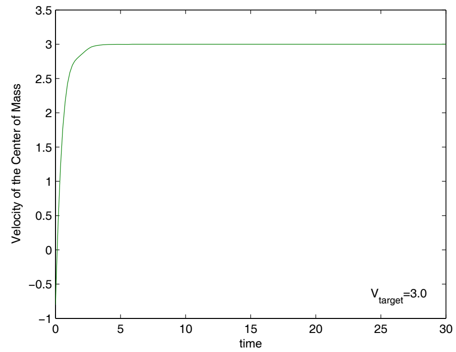
In both Figures 1 and 5, (a) shows the time history of the velocity of the center of the mass, *i.e.*, $v_{c.m.}$, (b) shows the error states of the center of the mass, *i.e.*, e_{1av} and e_{2av}, and (c) shows the control history.

2 Adding Surface Vibration

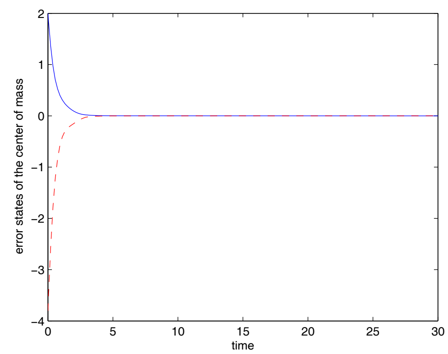
2.1 The Proposed Problem

With additions of normal oscillations to the substrate, the equation is given by:

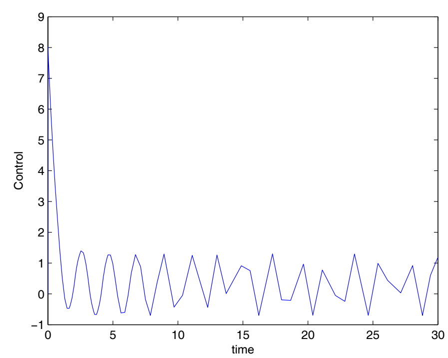
$$\begin{aligned}\ddot{\phi}_i + \gamma\dot{\phi}_i + (1 + A\sin(\omega t))\sin(\phi_i) &= \alpha(vt - \phi_{cm})\\ &+ F_i(\phi_i, \phi_j)\end{aligned} \tag{24}$$



(a)



(b)



(c)

Fig. 1. Tracking performance of the average system for targeted value $v_{target} = 3$: (a) the time history of the velocity of the center of the mass, (b) the time history of the error states of the center of the mass with the solid line denoting e_{1av} and the dashed line denoting e_{2av}, (c) the control history.

Using the average error states as defined in Subsection 1.2, we have the state space model for normal oscillation:

$$\begin{aligned}\dot{e}_{1av} &= e_{2av}\\ \dot{e}_{2av} &= -[1 + A\sin(\omega t)]\frac{1}{N}\sum_{i=1}^{N}\sin(e_{i1} + v_0 t) - \gamma e_{2av}\\ &\quad -\alpha e_{1av} - \gamma v_0\end{aligned} \tag{25}$$

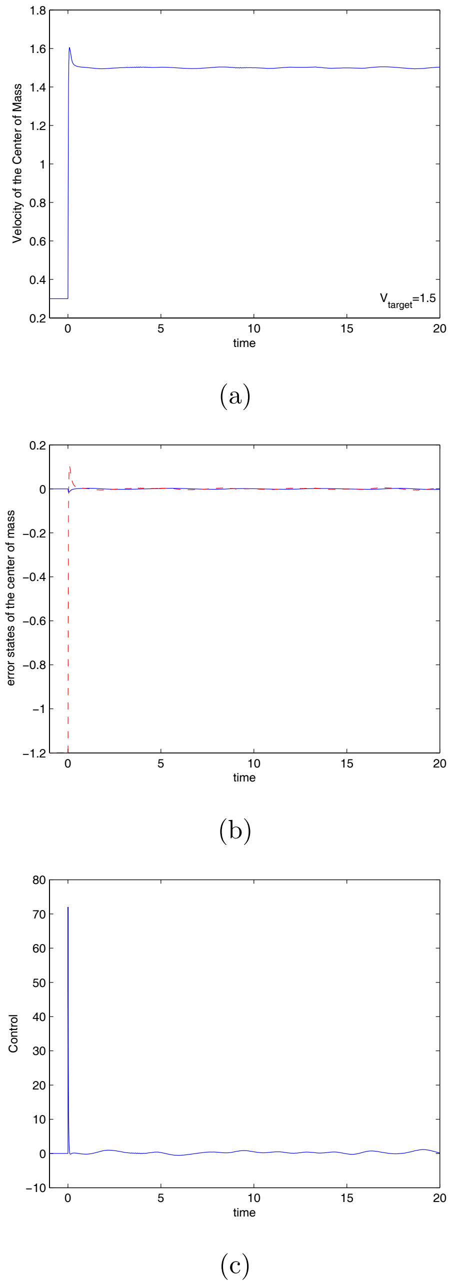


Fig. 2. Tracking performance of the average system for targeted value $v_{target} = 1.5$: (a) the time history of the velocity of the center of the mass, (b) the time history of the error states of the center of the mass with the solid line denoting e_{1av} and the dashed line denoting e_{2av}, (c) the control history.

Investigate one or more of the following issues:

- Investigate stability property of (25);
- Quantify transient response of the oscillating system using analytic methods;
- Establish the parameters of oscillation (A and ω) to ensure stability and desired transient responses.
- Simulate the system and verify the theoretical findings.

2.2 *Suggested Readings*

Khalil's Nonlinear Systems: Chapter 10 on periodic perturbation and averaging.