Cooperative Stabilization and Tracking for Linear Dynamic Systems

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Summary. A key problem in cooperative control is the convergence to a common value of the systems, which is called the consensus or agreement problem. Among recent results in cooperative control of multi-agent dynamic systems, the method used can be categorized in either graph theory or matrix theory based techniques, and the agent dynamics is either a single/double integrator or a linear system transformable to a canonical form. We design cooperative stabilization and tracking control in the framework of Lyapunov theorem for general linear dynamic systems. Decentralized control laws are explicitly constructed for individual systems with inter-system communications. Simulations show asymptotically tracking of a time-varying trajectory with pre-designated formation for a group of dynamic agents.

1 Introduction

Cooperative control has been an active research area due to its application importance in robotics, networked systems, and biological systems. A key problem in cooperative control is the convergence to a common value of the systems, which is called the consensus or agreement problem. Since Jadbabaie, Lin and Morse[1] present the first analytic results for analyzing cooperative behaviors of networked agents, excellent work has appeared in the literature[8]. It is revealed[2, 4, 9] that the sufficient and necessary condition for a networked system to achieve consensus is that the information exchange topology has a spanning tree. Olfati-Saber and Murray[5] relate information flow and communication structure to the stability property of the group of agents. Local control laws are presented to achieve formation stability[10, 3]. Recently, Qu[6, 7] presents a comprehensive study of consensus problem using matrix-theory-based framework, where cooperative control for dynamic vehicles in their canonical forms is solved explicitly.

Among existing cooperative control results, most focuses on single or double integrator dynamics or a particular vehicle dynamics, except that Qu[6, 7] provides solutions to general, higher dimensional linear systems which are in
their canonical forms. Though the dynamics of many agent systems can be transformed into one of the above structures, the extension to general linear dynamic systems is needed from both theoretical and practical points of view. From another perspective, the method used in most existing consensus work is either graph-theory or matrix theory, and the core interest has been the convergence of consensus protocol in association with the communication topology. While such problem is well understood, the problem of applying the consensus principle to general dynamic systems is naturally the next to explore.

We present cooperative stabilization and tracking control in the paper. A group of dynamic systems is designed to cooperatively get to their equilibrium points or to track a set of desired trajectories. Lyapunov theorem based methods are used in proving the convergence of the system state or the error (between the system and its reference). Decentralized feedback control laws for individual systems are explicitly constructed with inter-system communications. Simulation results are provided which demonstrate satisfactory performances. Besides the formation stability demonstrated, the method used in the paper sets up a connection between Lyapunov stability with consensus protocol, and provides a theoretical tool to study cooperative control for uncertain dynamic systems or systems with uncertain communication channels.

2 Problem Statement

We assume the $i$th vehicle has the following linear model:

$$\dot{x}_i = A_i x_i + B_i u_i$$

where $x_i \in \mathbb{R}^m$, $u_i \in \mathbb{R}^k$, $A_i \in \mathbb{R}^{m \times m}$, and $B_i \in \mathbb{R}^{m \times 1}$.

For a group of vehicles $i = 1, \ldots, N$ to achieve cooperative behaviors such as formation, we need to design control law

$$u_i = u_i(x_i, x_j), \quad j \in \mathcal{N}_i$$

where $x_j$ is the state of the $j$th vehicle, and $\mathcal{N}_i$ denotes the set of the neighbors of the $i$th vehicle. It can be seen that the control law is decentralized in the sense that it uses feedback from its own and its neighbors’ states only.

Depending on the control objectives, we define the cooperative stabilization and cooperative tracking control problems as follows:

**Cooperative Stabilization:** For a group of multi-vehicle systems (1) with $i = 1, \ldots, N$, given communication structure of the group forming a connected undirected graph, design decentralized control law

$$u_i = u_i(x_i, x_j), \quad j \in \mathcal{N}_i,$$

such that the vehicles return to their own equilibriums and keep formation.

That is,
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\[ x_i = 0 \quad \text{as} \quad t \to \infty \]
\[ x_i = x_j \quad \text{as} \quad t \to \infty \]  
(4)

for all \( i, j = 1, 2, \ldots, N \).

**Cooperative Tracking:** Given a set of formation trajectories described by:

\[ \dot{x}_{id} = A_{id} x_{id} + B_{id} u_{id}, \]  
(5)

design decentralized control law

\[ u_i = u_i(x_i, x_j) \quad j \in \mathcal{N}_i, \]  
(6)

such that the \( i \)-th vehicle asymptotically tracks its reference trajectory and keep formation. That is,

\[ e_i = 0 \quad \text{as} \quad t \to \infty \]
\[ e_i = e_j \quad \text{as} \quad t \to \infty \]  
(7)

for all \( i, j = 1, 2, \ldots, N \), where \( e_i = x_i - x_{id} \).

3 Preliminaries from Graph and Matrix Theory

We use some notations from graph theory to describe the communications structure of the group. A graph consists of a pair \((V, E)\), where \(V\) is a nonempty set of nodes and \(E \subseteq V^2\) is a set of pairs of nodes, called edges. \(E\) is unordered (or ordered) for an undirected (or directed) graph. A (undirected or directed) path is a sequence of (unordered or ordered) edges connecting two distinct vertices. A graph is called connected if there is a path between any distinct pair of nodes. A tree is a graph where every node, except the root, has exactly one parent node. A spanning tree is a tree formed by graph edges that connect all the nodes of the graph. A graph has a spanning tree if there exists a spanning tree that is a subset of the graph.

We use the Laplacian matrix, \(L_G\), to describe the connectivity of the nodes in a graph. \(L_G = D - A\), where \(A\) is the adjacent matrix with diagonal entries 0 and off-diagonal entries \(a_{ij} = 1\) if \((j, i) \in E\); \(D\) is the degree matrix with diagonal entries \(d_{ii} = \{j \in V : (j, i) \in E\}\) and off-diagonal entries 0. By definition, \(L_G\) is a zero row sum matrix. \(L_G\) is positive semi-definite for an undirected graph.

**Definition 1.** A matrix \(E \in \mathbb{R}^{r \times r}\) is said to be reducible if the set of its indices, \(I \triangleq \{1, 2, \ldots, r\}\), can be divided into two disjoint nonempty set \(S \triangleq \{i_1, i_2, \ldots, i_\mu\}\) and \(S' \triangleq T \setminus S = \{i_1, i_2, \ldots, i_\beta\}\) (with \(\mu + \nu = r\)) such that \(e_{i_\alpha, i_\beta} = 0\), where \(\alpha = 1, \ldots, \mu\) and \(\beta = 1, \ldots, \nu\). A matrix is said to be irreducible if it is not reducible.
It is easy to see that the Laplacian matrix of a connected undirected graph
is irreducible since its adjacent matrix does not have two disjoint vertex sets.

We’ll need the following properties which was proposed by Wu [12, 11] for
zero row sum matrices.

**Lemma 1.** Let the set $W$ consists all zero row sum matrices which have only
non-positive off-diagonal elements. If $A$ is a symmetric matrix in $W$, then
$A$ can be decomposed as $A = M^T M$ where $M$ is a matrix such that row
$i$ consists of zeros and exactly one entry $\alpha_i$ and one entry $-\alpha_i$ for some
nonzero $\alpha_i$. Furthermore, if $A$ is irreducible, then the graph associated with
$M$ is connected.

The matrix $M$ can be constructed as follows ([11]): For each nonzero row
of $A$, we generate several rows of $M$ if the same length: for the
$i$th row of $A$, and for each $i < j$ such that $A_{ij} = -\alpha$ for some $\alpha > 0$, we add a row to $M
$ with the $i$th element being $\sqrt{\alpha}$, and the $j$th element being $-\sqrt{\alpha}$. This matrix
$M$ satisfies $A = M^T M$.

### 4 Cooperative Control Design

#### 4.1 Cooperative Stabilization for One Dimensional Vehicle Model

First, we illustrate the idea of the design using a one-dimension vehicle system. The
$i$th vehicle system model is:

$$\dot{x}_i = -x_i + u_i \quad (8)$$

where $x_i, u_i$ are scalars. It’s obvious that the equilibrium of each vehicle is at 0. We show next that the consensus protocol,

$$u_i = -\sum_{j \in N_i} \alpha_{ij} (x_i - x_j), \quad (9)$$

achieves cooperative stabilization using Lyapunov theory.

Define Lyapunov candidate

$$V = \sum_{i=1}^{N} \frac{1}{2} x_i^2. \quad (10)$$

Its time derivative along the system dynamics is

$$\dot{V} = \sum_{i=1}^{N} (-x_i^2 + x_i u_i)$$

$$= -\sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} x_i \sum_{j \in N_i} \alpha_{ij} (x_i - x_j)$$

$$= -\|x\|^2 - x^T L x \quad (11)$$
where $x = [x_1, x_2, \ldots, x_N]^T$, and $L$ is the positive semi-definite Laplacian matrix defined in Section 3.

From (11), we obtain:

$$\dot{V} \leq -\|x\|^2$$

which is negative definite. From Lyapunov theory, we conclude that each vehicle system goes to its equilibrium, 0, as $t \to \infty$.

From (11), we also obtain:

$$\dot{V} \leq -x^T L x = - \sum_{i,j \in \mathcal{N}_i} a_{ij} (x_i - x_j)^2$$

where $a_{ij}$ are real constants. From LaSalle's invariance theorem, we know that the system gets to the invariance set $\sum_{i,j} a_{ij} (x_i - x_j)^2 = 0$. Since the graph is connected, we get $x_i = x_j$ for all $i, j$ as $t \to \infty$.

4.2 Cooperative Stabilization for Higher Dimensional Vehicle Model

In this subsection, we take the vehicle model as the general linear form in (1). First, rewrite the system dynamics in the following compact form:

$$\dot{x} = Ax + Bu$$

where

$$x = [x_1^T, x_2^T, \ldots, x_N^T]^T, \quad A = I_N \otimes A_i, \quad B = I_N \otimes B_i,$$

and $I_N$ is the N-dimensional identity matrix.

Choose Lyapunov candidate

$$V = x^T P x$$

where

$$P = I_N \otimes P_i$$

and $P_i$ are positive definite matrices to be chosen later.

Taking time derivatives of $V$ along the vehicles’ dynamics, we get

$$\dot{V} = x^T (PA + A^T P)x + x^T PBu.$$  

Let

$$u = u_i + u_j.$$  

That is, we divide the control into two separate parts: one for stabilizing its own states, $u_i$, called stabilizing control law; the other for the group coordination, $u_j$, called coordination control law.
We choose the stabilizing control law as
\[ u_i = -(I_N \otimes \epsilon_i B_i^T P_i)x \] (18)
where \( \epsilon_i \) is a positive constant. Substituting it into (16), we get
\[ \dot{V} = \sum_{i=1}^{N} \{ x_i^T (P_i A_i + A_i^T P_i - 2\epsilon_i P_i B_i B_i^T P_i) x_i \} + 2x^T P Bu_j. \] (19)
Choose \( P_i \) to solve the following algebraic Riccati equation
\[ P_i A_i + A_i^T P_i - 2\epsilon_i P_i B_i B_i^T P_i + Q_i = 0 \] (20)
where \( Q_i \) is a positive definite matrix. Then we obtain
\[ \dot{V} = -x^T Q x + 2x^T P Bu_j \] (21)
where
\[ Q = I_N \otimes Q_i. \]

To design the coordination control law, we need to make use of the Laplacian matrix of the group, \( L_G \). Choose
\[ u_j = -F L x \] (22)
where
\[ F = I_N \otimes F_i, \quad L = L_G \otimes I_m, \]
and \( F_i \in \mathbb{R}^{1 \times m} \) to be chosen later. Recall that \( m \) is the dimension of the vehicle state \( x_i \). Simplify \( BFL \) as follows:
\[ BFL = (I_N \otimes B_i)(I_N \otimes F_i)(L_G \otimes I_m) \]
\[ = (I_N \otimes B_i F_i)(L_G \otimes I_m) \]
\[ = L_G \otimes B_i F_i. \] (23)
Substituting \( u_j \) into the second term of (21), we have
\[ x^T P Bu_j = -x^T P BFL x \]
\[ = -x^T (I_n \otimes P_i)(L_G \otimes I_m)x \]
\[ = -x^T [L_G \otimes (P_i B_i F_i)]x \] (24)
From Lemma 1, we know that \( L_G \) can be decomposed as \( L_G = C^T C \) where \( C \) is a matrix such that row \( i \) consists of zeros and exactly one entry \( \alpha_i \) and one entry \( -\alpha_i \) for some nonzero \( \alpha_i \). Therefore we have
\[ x^T P Bu_j = -x^T [(C^T C \otimes (P_i B_i F_i)]x \]
\[ = -x^T (C^T \otimes I_m)(C \otimes (P_i B_i F_i)]x \]
\[ = -\sum_{i,j} \alpha_i^2 (x_i - x_j)^T (P_i B_i F_i)(x_i - x_j) \] (25)
Since the graph associated with $C$ is connected, all $i, j = 1, 2, \ldots, N$ are included in the above equation.

Substitute the above equation into (21), we obtain

$$
\dot{V} = -x^TQx - \sum_{i,j} 2\alpha_{ij}^2(x_i - x_j)^T(P_i B_i F_i)(x_i - x_j)
$$

(26)

Choose $F_i$ such that $P_i B_i F_i$ is positive definite, then the second term of the above equation is negative definite. We conclude stability for each individual vehicle since

$$
\dot{V} \leq -x^TQx.
$$

(27)

Also,

$$
\dot{V} \leq - \sum_{i,j} 2\alpha_{ij}^2(x_i - x_j)^T(P_i B_i F_i)(x_i - x_j)
$$

(28)

The right hand side of the equation is 0 only when $x_i = x_j$ for all $i, j$. From the LaSalle’s invariant theorem, we conclude that the group achieves cooperative stabilization.

Note that if $P_i B_i F_i$ is positive semi-definite, which is a weaker condition, then $x_i - x_j$ does not guarantee to go to zero but a small neighborhood of zero by (28).

We have the following theorem to summarize the result of cooperative stabilization:

**Theorem 1.** Given a strongly connected communication graph, the cooperative stabilization problem is solvable using the decentralized control law (17), (18), (22) if there exists positive definite matrices $P_i, Q_i$ to solve (20), and a feedback control matrix, $F_i$, such that $P_i B_i F_i$ is positive definite.

### 4.3 Cooperative Tracking

Define the error state as follows:

$$
e_i = x_i - x_{id}
$$

(29)

We have the error dynamics

$$
\dot{e}_i = A_i e_i + B_i(u_i - u_{id}).
$$

(30)

Rewrite it in the compact form

$$
\dot{E} = AE + BV
$$

(31)

where
\[ A = I_N \otimes A_i, \quad B = I_N \otimes B_i, \]
\[ E = [e_1^T, e_2^T, \ldots, e_N^T]^T, \quad V = [v_1^T, v_2^T, \ldots, v_N^T]^T \]

and \( v_i = u_i - u_{id} \) for \( i = 1, 2, \ldots, N \).

Now the cooperative tracking design for the original system (1) turns into cooperative stabilization design for the error system (30). Since (31) is in the same form as (14), it follows the same procedure to design \( v_i \) as that described in 4.2. After we obtain decentralized control \( v_i \), we can easily get \( u_i = v_i + u_{id} \).

In the case that we do not have the reference model (5), but only a reference trajectory \( x_{id} \) and its derivatives \( \dot{x}_{id} \), we can define the dynamics of the error systems as

\[
\dot{e}_i = A_i e_i + (A x_{id} + B_i u_i - \dot{x}_{id}) \overset{\text{def}}{=} A_i e_i + v_i \tag{32}
\]

By defining the new virtual control input \( v_i \), the cooperative tracking problem is solved.

## 5 Simulations

Consider the following two-dimensional vehicle model:

\[
\begin{align*}
x_{i1} &= x_{i2} + u_{i1} \\
x_{i2} &= u_{i2}
\end{align*} \tag{33}
\]

where \( i = 1, 2, 3, 4 \). Correspondingly,

\[
\begin{align*}
A_i &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\end{align*}
\]

The communication structure of the vehicles is shown as in Figure 1. Its Laplacian matrix is

\[
L_G = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix} \tag{34}
\]

Choose \( Q_i = I_2 \), i.e., a 2-dimensional identity matrix, and \( \epsilon_i = 0.5 \). The solution to the Riccati equation (20) is

\[
P_i = \begin{bmatrix} 0.9102 & 0.4142 \\ 0.4142 & 1.28720 \end{bmatrix}.
\]

The decentralized control law is:

\[ u_i = -\epsilon_i B_i^T P_i x_i - (FLx)_i, \tag{35} \]
Fig. 1. Communication topology of a four-robot team

Fig. 2. The time history of the first states of four robots

Fig. 3. The time history of the second states of four robots
where \((FLx)_i\) denotes the \(i\)th row element of \(FLx\). The simulation results are shown in Figures 2-3.

To demonstrate the performance of cooperative tracking, we show the states of the four-robot team following a unit circle in a rectangular formation. The geometry of the formation is illustrated in Figure 4. The origin of the formation frame tracks a unit circle. To set up the reference trajectories appropriately, we define a set of coordinations, \(d_i, i = 1, 2, \ldots, N\), on a moving frame \((h_1, h_2)\), see Figure 5. The reference trajectory is:

\[
x_i^d = P_i + d_{i1}h_1 + d_{i2}h_2
\]

where \(P_i = [\cos t \ \sin t]^T\), \(h_1 = [-\sin t \ \cos t]^T\), \(h_2 = [\cos t \ \sin t]^T\), and \(d_i, i = 1, 2, 3, 4\) are given in Figure 4.

Define the error states \(e_i = x_i - x_i^d\). Then

\[
\dot{e}_{i1} = e_{i2} + v_{i1}
\]
\[
e_{i2} = v_{i2}
\]

where

\[
v_{i1} = u_{i1} + x_{i2}^d - \dot{x}_{i1}^d
\]
\[
v_{i2} = u_{i2} - \dot{x}_{i2}^d.
\]

We design the new control input \((v_{i1}, v_{i2})\) following the procedure described in Section 4.3. The state histories of the four robots are shown in Figure 6. It can be seen that cooperative formation with formation is achieved.

6 Conclusions

We have designed a cooperative stabilization and a cooperative tracking control for a group of dynamic linear systems in their general state space representations. Decentralized control laws are explicitly constructed for each system.
Fig. 5. Formation of a three-robot team in a moving coordinate frame

Fig. 6. The time history of four robots tracking a unit circle with rigid formation

with information exchange between them. Making use of the existing results on consensus protocol, we combined the vehicle-level control and rendered the group tracking a desired time-varying trajectory with pre-designate formation. Lyapunov theorem based method is used in deriving the control laws, and simulation results are illustrated for a four-robot group circling a unit circle. Future research includes the study of cooperative control for uncertain dynamic systems and systems with uncertain communication channels under the present framework.

References


